CHAPTER 15

GENERAL THEORY OF CYLINDRICAL SHELLS

114. A Circular Cylindrical Shell Loaded Symmetrically with Respect to Its Axis. In practical applications we frequently encounter problems in which a circular cylindrical shell is submitted to the action of forces distributed symmetrically with respect to the axis of the cylinder. The stress distribution in cylindrical boilers submitted to the action of steam pressure, stresses in cylindrical containers having a vertical axis and submitted to internal liquid pressure, and stresses in circular pipes under uniform internal pressure are examples of such problems.

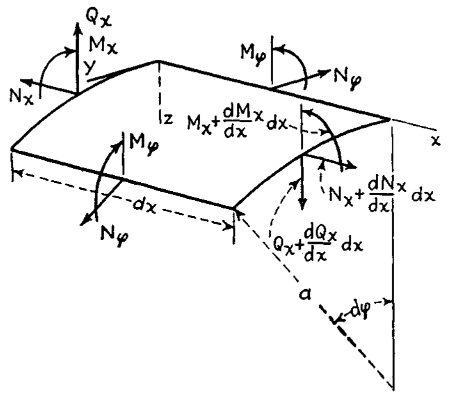

FIG. 235

To establish the equations required for the solution of these problems we consider an element, as shown in Figs. 228*a* and 235, and consider the equations of equilibrium. It can be concluded from symmetry that the membrane shearing forces $N_{x\varphi} = N_{\varphi x}$ vanish in this case and that forces N_φ are constant along the circumference. Regarding the transverse shearing forces, it can also be concluded from symmetry that only the forces Q_x do not vanish. Considering the moments acting on the element in Fig. 235, we also conclude from symmetry that the twisting moments $M_{x\varphi} = M_{\varphi x}$ vanish and that the bending moments M_φ are constant along the circumference. Under such conditions of symmetry three of the six equations of equilibrium of the element are identically satisfied, and we have to consider only the remaining three equations, *viz.*, those obtained by projecting the forces on the x and z axes and by taking the moment of the forces about the y axis. Assuming that the external forces consist only of a pressure normal to the surface, these three equations of equilibrium are

$$\begin{aligned} \frac{dN_x}{dx} a\,dx\,d\varphi &= 0 \\ \frac{dQ_x}{dx} a\,dx\,d\varphi + N_\varphi\,dx\,d\varphi + Za\,dx\,d\varphi &= 0 \\ \frac{dM_x}{dx} a\,dx\,d\varphi - Q_x a\,dx\,d\varphi &= 0 \end{aligned} \tag{a}$$

The first one indicates that the forces N_x are constant,[1] and we take them equal to zero in our further discussion. If they are different from zero, the deformation and stress corresponding to such constant forces can be easily calculated and superposed on stresses and deformations produced by lateral load. The remaining two equations can be written in the following simplified form:

$$\begin{aligned} \frac{dQ_x}{dx} + \frac{1}{a} N_\varphi &= -Z \\ \frac{dM_x}{dx} - Q_x &= 0 \end{aligned} \tag{b}$$

These two equations contain three unknown quantities: N_φ, Q_x, and M_x. To solve the problem we must therefore consider the displacements of points in the middle surface of the shell.

From symmetry we conclude that the component v of the displacement in the circumferential direction vanishes. We thus have to consider only the components u and w in the x and z directions, respectively. The expressions for the strain components then become

$$\epsilon_x = \frac{du}{dx} \qquad \epsilon_\varphi = -\frac{w}{a} \tag{c}$$

Hence, by applying Hooke's law, we obtain

$$\begin{aligned} N_x &= \frac{Eh}{1-\nu^2}(\epsilon_x + \nu\epsilon_\varphi) = \frac{Eh}{1-\nu^2}\left(\frac{du}{dx} - \nu\frac{w}{a}\right) = 0 \\ N_\varphi &= \frac{Eh}{1-\nu^2}(\epsilon_\varphi + \nu\epsilon_x) = \frac{Eh}{1-\nu^2}\left(-\frac{w}{a} + \nu\frac{du}{dx}\right) \end{aligned} \tag{d}$$

From the first of these equations it follows that

$$\frac{du}{dx} = \nu\frac{w}{a}$$

[1] The effect of these forces on bending is neglected in this discussion.

and the second equation gives

$$N_\varphi = -\frac{Ehw}{a} \tag{e}$$

Considering the bending moments, we conclude from symmetry that there is no change in curvature in the circumferential direction. The curvature in the x direction is equal to $-d^2w/dx^2$. Using the same equations as for plates, we then obtain

$$\begin{aligned} M_\varphi &= \nu M_x \\ M_x &= -D\frac{d^2w}{dx^2} \end{aligned} \tag{f}$$

where

$$D = \frac{Eh^3}{12(1-\nu^3)}$$

is the flexural rigidity of the shell.

Returning now to Eqs. (*b*) and eliminating Q_x from these equations, we obtain

$$\frac{d^2M_x}{dx^2} + \frac{1}{a}N_\varphi = -Z$$

from which, by using Eqs. (*e*) and (*f*), we obtain

$$\frac{d^2}{dx^2}\left(D\frac{d^2w}{dx^2}\right) + \frac{Eh}{a^2}w = Z \tag{273}$$

All problems of symmetrical deformation of circular cylindrical shells thus reduce to the integration of Eq. (273).

The simplest application of this equation is obtained when the thickness of the shell is constant. Under such conditions Eq. (273) becomes

$$D\frac{d^4w}{dx^4} + \frac{Eh}{a^2}w = Z \tag{274}$$

Using the notation

$$\beta^4 = \frac{Eh}{4a^2D} = \frac{3(1-\nu^2)}{a^2h^2} \tag{275}$$

Eq. (274) can be represented in the simplified form

$$\frac{d^4w}{dx^4} + 4\beta^4w = \frac{Z}{D} \tag{276}$$

This is the same equation as is obtained for a prismatical bar with a flexural rigidity D, supported by a continuous elastic foundation and submitted to the action of a load of intensity Z.* The general solution of this equation is

$$w = e^{\beta x}(C_1\cos\beta x + C_2\sin\beta x) + e^{-\beta x}(C_3\cos\beta x + C_4\sin\beta x) + f(x) \tag{277}$$

* See S. Timoshenko, "Strength of Materials," part II, 3d ed., p. 2, 1956.

in which $f(x)$ is a particular solution of Eq. (276), and $C_1, \ldots, C_4$ are the constants of integration which must be determined in each particular case from the conditions at the ends of the cylinder.

Take, as an example, a long circular pipe submitted to the action of bending moments M_0 and shearing forces Q_0, both uniformly distributed along the edge $x = 0$ (Fig. 236). In this case there is no pressure Z distributed over the surface of the shell, and $f(x) = 0$ in the general solution (277). Since the forces applied at the end $x = 0$ produce a local bending which dies out rapidly as the distance x from the loaded end increases, we conclude that the first term on the right-hand side of Eq. (277) must vanish.[1] Hence, $C_1 = C_2 = 0$, and we obtain

$$w = e^{-\beta x}(C_3 \cos \beta x + C_4 \sin \beta x) \tag{g}$$

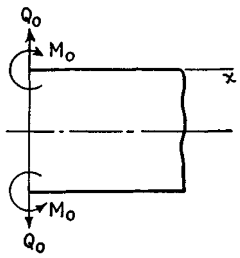

FIG. 236

The two constants C_3 and C_4 can now be determined from the conditions at the loaded end, which may be written

$$\begin{aligned} (M_x)_{x=0} &= -D\left(\frac{d^2w}{dx^2}\right)_{x=0} = M_0 \\ (Q_x)_{x=0} &= \left(\frac{dM_x}{dx}\right)_{x=0} = -D\left(\frac{d^3w}{dx^3}\right)_{x=0} = Q_0 \end{aligned} \tag{h}$$

Substituting expression (g) for w, we obtain from these end conditions

$$C_3 = -\frac{1}{2\beta^3 D}(Q_0 + \beta M_0) \qquad C_4 = \frac{M_0}{2\beta^2 D} \tag{i}$$

Thus the final expression for w is

$$w = \frac{e^{-\beta x}}{2\beta^3 D}[\beta M_0(\sin \beta x - \cos \beta x) - Q_0 \cos \beta x] \tag{278}$$

The maximum deflection is obtained at the loaded end, where

$$(w)_{x=0} = -\frac{1}{2\beta^3 D}(\beta M_0 + Q_0) \tag{279}$$

The negative sign for this deflection results from the fact that w is taken positive toward the axis of the cylinder. The slope at the loaded end is

[1] Observing the fact that the system of forces applied at the end of the pipe is a balanced one and that the length of the pipe may be increased at will, this follows also from the principle of Saint-Venant; see, for example, S. Timoshenko and J. N. Goodier, "Theory of Elasticity," 2d ed., p. 33, 1951.

obtained by differentiating expression (278). This gives

$$\left(\frac{dw}{dx}\right)_{x=0} = \frac{e^{-\beta x}}{2\beta^2 D}\left[2\beta M_0 \cos\beta x + Q_0(\cos\beta x + \sin\beta x)\right]_{x=0}$$
$$= \frac{1}{2\beta^2 D}(2\beta M_0 + Q_0) \qquad (280)$$

By introducing the notation

$$\begin{aligned} \varphi(\beta x) &= e^{-\beta x}(\cos\beta x + \sin\beta x) \\ \psi(\beta x) &= e^{-\beta x}(\cos\beta x - \sin\beta x) \\ \theta(\beta x) &= e^{-\beta x}\cos\beta x \\ \zeta(\beta x) &= e^{-\beta x}\sin\beta x \end{aligned} \qquad (281)$$

the expressions for deflection and its consecutive derivatives can be represented in the following simplified form:

$$\begin{aligned} w &= -\frac{1}{2\beta^3 D}\left[\beta M_0 \psi(\beta x) + Q_0\theta(\beta x)\right] \\ \frac{dw}{dx} &= \frac{1}{2\beta^2 D}\left[2\beta M_0\theta(\beta x) + Q_0\varphi(\beta x)\right] \\ \frac{d^2w}{dx^2} &= -\frac{1}{2\beta D}\left[2\beta M_0\varphi(\beta x) + 2Q_0\zeta(\beta x)\right] \\ \frac{d^3w}{dx^3} &= \frac{1}{D}\left[2\beta M_0\zeta(\beta x) - Q_0\psi(\beta x)\right] \end{aligned} \qquad (282)$$

The numerical values of the functions $\varphi(\beta x)$, $\psi(\beta x)$, $\theta(\beta x)$, and $\zeta(\beta x)$ are given in Table 84.[1] The functions $\varphi(\beta x)$ and $\psi(\beta x)$ are represented graphically in Fig. 237. It is seen from these curves and from Table 84

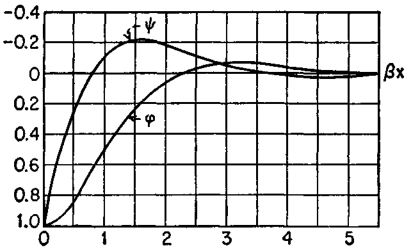

FIG. 237

that the functions defining the bending of the shell approach zero as the quantity βx becomes large. This indicates that the bending produced in the shell is of a local character, as was already mentioned at the beginning when the constants of integration were calculated.

If the moment M_x and the deflection w are found from expressions

[1] The figures in this table are taken from the book by H. Zimmermann, "Die Berechnung des Eisenbahnoberbaues," Berlin, 1888.

(282), the bending moment M_φ is obtained from the first of the equations (*f*), and the value of the force N_φ from Eq. (*e*). Thus all necessary information for calculating stresses in the shell can be found.

115. Particular Cases of Symmetrical Deformation of Circular Cylindrical Shells. *Bending of a Long Cylindrical Shell by a Load Uniformly Distributed along a Circular Section* (Fig. 238). If the load is far enough from the ends of the cylinder, solution (278) can be used for each half of

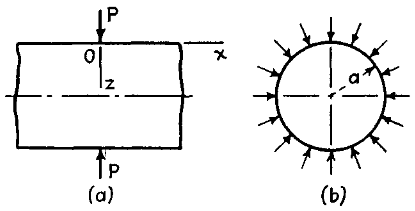

FIG. 238

the shell. From considerations of symmetry we conclude that the value of Q_0 in this case is $-P/2$. We thus obtain for the right-hand portion

$$w = \frac{e^{-\beta x}}{2\beta^3 D}\left[\beta M_0(\sin\beta x - \cos\beta x) + \frac{P}{2}\cos\beta x\right] \qquad (a)$$

where x is measured from the cross section at which the load is applied. To calculate the moment M_0 which appears in expression (*a*) we use expression (280), which gives the slope at $x = 0$. In our case this slope vanishes because of symmetry. Hence,

$$2\beta M_0 - \frac{P}{2} = 0$$

and we obtain

$$M_0 = \frac{P}{4\beta} \qquad (b)$$

Substituting this value in expression (*a*), the deflection of the shell becomes

$$w = \frac{Pe^{-\beta x}}{8\beta^3 D}(\sin\beta x + \cos\beta x) = \frac{P}{8\beta^3 D}\varphi(\beta x) \qquad (283)$$

and by differentiation we find

$$\begin{aligned}
\frac{dw}{dx} &= -2\beta\frac{P}{8\beta^3 D}e^{-\beta x}\sin\beta x = -\frac{P}{4\beta^2 D}\zeta(\beta x)\\
\frac{d^2w}{dx^2} &= 2\beta^2\frac{P}{8\beta^3 D}\cdot e^{-\beta x}(\sin\beta x - \cos\beta x) = -\frac{P}{4\beta D}\psi(\beta x)\\
\frac{d^3w}{dx^3} &= 4\beta^3\frac{P}{8\beta^3 D}e^{-\beta x}\cos\beta x = \frac{P}{2D}\theta(\beta x)
\end{aligned} \qquad (c)$$

TABLE 84. TABLE OF FUNCTIONS φ, ψ, θ, AND ζ

βx	φ	ψ	θ	ζ
0	1.0000	1.0000	1.0000	0
0.1	0.9907	0.8100	0.9003	0.0903
0.2	0.9651	0.6398	0.8024	0.1627
0.3	0.9267	0.4888	0.7077	0.2189
0.4	0.8784	0.3564	0.6174	0.2610
0.5	0.8231	0.2415	0.5323	0.2908
0.6	0.7628	0.1431	0.4530	0.3099
0.7	0.6997	0.0599	0.3798	0.3199
0.8	0.6354	−0.0093	0.3131	0.3223
0.9	0.5712	−0.0657	0.2527	0.3185
1.0	0.5083	−0.1108	0.1988	0.3096
1.1	0.4476	−0.1457	0.1510	0.2967
1.2	0.3899	−0.1716	0.1091	0.2807
1.3	0.3355	−0.1897	0.0729	0.2626
1.4	0.2849	−0.2011	0.0419	0.2430
1.5	0.2384	−0.2068	0.0158	0.2226
1.6	0.1959	−0.2077	−0.0059	0.2018
1.7	0.1576	−0.2047	−0.0235	0.1812
1.8	0.1234	−0.1985	−0.0376	0.1610
1.9	0.0932	−0.1899	−0.0484	0.1415
2.0	0.0667	−0.1794	−0.0563	0.1230
2.1	0.0439	−0.1675	−0.0618	0.1057
2.2	0.0244	−0.1548	−0.0652	0.0895
2.3	0.0080	−0.1416	−0.0668	0.0748
2.4	−0.0056	−0.1282	−0.0669	0.0613
2.5	−0.0166	−0.1149	−0.0658	0.0492
2.6	−0.0254	−0.1019	−0.0636	0.0383
2.7	−0.0320	−0.0895	−0.0608	0.0287
2.8	−0.0369	−0.0777	−0.0573	0.0204
2.9	−0.0403	−0.0666	−0.0534	0.0132
3.0	−0.0423	−0.0563	−0.0493	0.0071
3.1	−0.0431	−0.0469	−0.0450	0.0019
3.2	−0.0431	−0.0383	−0.0407	−0.0024
3.3	−0.0422	−0.0306	−0.0364	−0.0058
3.4	−0.0408	−0.0237	−0.0323	−0.0085
3.5	−0.0389	−0.0177	−0.0283	−0.0106
3.6	−0.0366	−0.0124	−0.0245	−0.0121
3.7	−0.0341	−0.0079	−0.0210	−0.0131
3.8	−0.0314	−0.0040	−0.0177	−0.0137
3.9	−0.0286	−0.0008	−0.0147	−0.0140

TABLE 84. TABLE OF FUNCTIONS φ, ψ, θ, AND ζ (*Continued*)

βx	φ	ψ	θ	ζ
4.0	−0.0258	0.0019	−0.0120	−0.0139
4.1	−0.0231	0.0040	−0.0095	−0.0136
4.2	−0.0204	0.0057	−0.0074	−0.0131
4.3	−0.0179	0.0070	−0.0054	−0.0125
4.4	−0.0155	0.0079	−0.0038	−0.0117
4.5	−0.0132	0.0085	−0.0023	−0.0108
4.6	−0.0111	0.0089	−0.0011	−0.0100
4.7	−0.0092	0.0090	0.0001	−0.0091
4.8	−0.0075	0.0089	0.0007	−0.0082
4.9	−0.0059	0.0087	0.0014	−0.0073
5.0	−0.0046	0.0084	0.0019	−0.0065
5.1	−0.0033	0.0080	0.0023	−0.0057
5.2	−0.0023	0.0075	0.0026	−0.0049
5.3	−0.0014	0.0069	0.0028	−0.0042
5.4	−0.0006	0.0064	0.0029	−0.0035
5.5	0.0000	0.0058	0.0029	−0.0029
5.6	0.0005	0.0052	0.0029	−0.0023
5.7	0.0010	0.0046	0.0028	−0.0018
5.8	0.0013	0.0041	0.0027	−0.0014
5.9	0.0015	0.0036	0.0026	−0.0010
6.0	0.0017	0.0031	0.0024	−0.0007
6.1	0.0018	0.0026	0.0022	−0.0004
6.2	0.0019	0.0022	0.0020	−0.0002
6.3	0.0019	0.0018	0.0018	+0.0001
6.4	0.0018	0.0015	0.0017	0.0003
6.5	0.0018	0.0012	0.0015	0.0004
6.6	0.0017	0.0009	0.0013	0.0005
6.7	0.0016	0.0006	0.0011	0.0006
6.8	0.0015	0.0004	0.0010	0.0006
6.9	0.0014	0.0002	0.0008	0.0006
7.0	0.0013	0.0001	0.0007	0.0006

Observing from Eqs. (*b*) and (*f*) of the preceding article that

$$M_x = -D\frac{d^2w}{dx^2} \qquad Q_x = -D\frac{d^3w}{dx^3}$$

we finally obtain the following expressions for the bending moment and shearing force:

$$M_x = \frac{P}{4\beta}\psi(\beta x) \qquad Q_x = -\frac{P}{2}\theta(\beta x) \tag{284}$$

The results obtained are all graphically represented in Fig. 239. It is seen that the maximum deflection is under the load P and that its value as given by Eq. (283) is

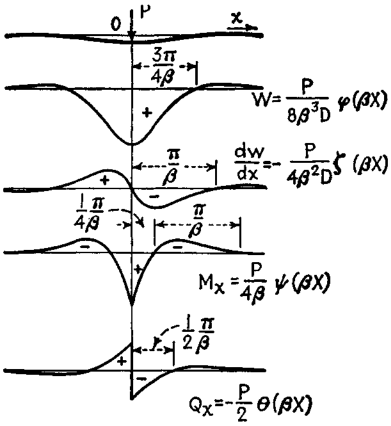

FIG. 239

$$w_{\max} = \frac{P}{8\beta^3 D} = \frac{Pa^2\beta}{2Eh} \tag{285}$$

The maximum bending moment is also under the load and is determined from Eq. (284) as

$$M_{\max} = \frac{P}{4\beta} \tag{286}$$

The maximum of the absolute value of the shearing force is evidently equal to $P/2$. The values of all these quantities at a certain distance from the load can be readily obtained by using Table 84. We see from this table and from Fig. 239 that all the quantities that determine the bending of the shell are small for $x > \pi/\beta$. This fact indicates that the bending is of a local character and that a shell of length $l = 2\pi/\beta$ loaded at the middle will have practically the same maximum deflection and the same maximum stress as a very long shell.

Having the solution of the problem for the case in which a load is concentrated at a circular cross section, we can readily solve the problem of a load distributed along a certain length of the cylinder by applying the principle of superposition. As an example let us consider the case of a load of intensity q uniformly distributed along a length l of a cylinder (Fig. 240). Assuming that the load is at a considerable distance from the ends of the cylinder, we can use solution (283) to calculate the deflections.

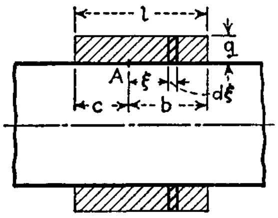

FIG. 240

The deflection at a point A produced by an elementary ring load of an intensity[1] $q\,d\xi$ at a distance ξ from A is obtained from expression (283) by substituting $q\,d\xi$ for P and ξ for x and is

$$\frac{q\,d\xi}{8\beta^3 D} e^{-\beta\xi}(\cos\beta\xi + \sin\beta\xi)$$

The deflection produced at A by the total load distributed over the

[1] $q\,d\xi$ is the load per unit length of circumference.

length l is then

$$w = \int_0^b \frac{q\,d\xi}{8\beta^3 D} e^{-\beta\xi}(\cos\beta\xi + \sin\beta\xi) + \int_0^c \frac{q\,d\xi}{8\beta^3 D} e^{-\beta\xi}(\cos\beta\xi + \sin\beta\xi)$$
$$= \frac{qa^2}{2Eh}(2 - e^{-\beta b}\cos\beta b - e^{-\beta c}\cos\beta c)$$

The bending moment at a point A can be calculated by similar application of the method of superposition.

Cylindrical Shell with a Uniform Internal Pressure (Fig. 241). If the edges of the shell are free, the internal pressure p produces only a hoop stress

$$\sigma_t = \frac{pa}{h}$$

and the radius of the cylinder increases by the amount

$$\delta = \frac{a\sigma_t}{E} = \frac{pa^2}{Eh} \tag{d}$$

If the ends of the shell are built in, as shown in Fig. 241*a*, they cannot move out, and local bending occurs at the edges. If the length l of the

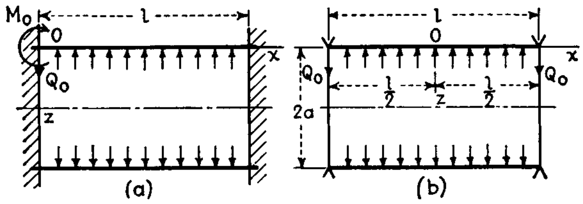

FIG. 241

shell is sufficiently large, we can use solution (278) to investigate this bending, the moment M_0 and the shearing force Q_0 being determined from the conditions that the deflection and the slope along the built-in edge $x = 0$ (Fig. 241*a*) vanish. According to these conditions, Eqs. (279) and (280) of the preceding article become

$$-\frac{1}{2\beta^3 D}(\beta M_0 + Q_0) = \delta$$
$$\frac{1}{2\beta^2 D}(2\beta M_0 + Q_0) = 0$$

where δ is given by Eq. (*d*).

Solving for M_0 and Q_0, we obtain

$$M_0 = 2\beta^2 D\delta = \frac{p}{2\beta^2} \qquad Q_0 = -4\beta^3 D\delta = -\frac{p}{\beta} \tag{287}$$

We thus obtain a positive bending moment and a negative shearing force acting as shown in Fig. 241*a*. Substituting these values in expressions (282), the deflection and the bending moment at any distance from the end can be readily calculated using Table 84.

If, instead of built-in edges, we have simply supported edges as shown in Fig. 241*b*, the deflection and the bending moment M_x vanish along the edge $M_0 = 0$, and we obtain, by using Eq. (279),

$$Q_0 = -2\beta^3 D\delta$$

By substituting these values in solution (278) the deflection at any distance from the end can be calculated.

It was assumed in the preceding discussion that the length of the shell is large. If this is not the case, the bending at one end cannot be considered as independent of the conditions at the other end, and recourse must be had to the general solution (277), which contains four constants of integration. The particular solution of Eq. (276) for the case of uniform load ($Z = -p$) is $-p/4\beta^4 D = -pa^2/Eh$. The general solution (277) can then be put in the following form by the introduction of hyperbolic functions in place of the exponential functions:

$$w = -\frac{pa^2}{Eh} + C_1 \sin\beta x \sinh\beta x + C_2 \sin\beta x \cosh\beta x + C_3 \cos\beta x \sinh\beta x + C_4 \cos\beta x \cosh\beta x \qquad (e)$$

If the origin of coordinates is taken at the middle of the cylinder, as shown in Fig. 241*b*, expression (*e*) must be an even function of x. Hence

$$C_2 = C_3 = 0 \qquad (f)$$

The constants C_1 and C_4 must now be selected so as to satisfy the conditions at the ends. If the ends are simply supported, the deflection and the bending moment M_x must vanish at the ends, and we obtain

$$(w)_{x=l/2} = 0 \qquad \left(\frac{d^2w}{dx^2}\right)_{x=l/2} = 0 \qquad (g)$$

Substituting expression (*e*) in these relations and remembering that $C_2 = C_3 = 0$, we find

$$\begin{aligned} -\frac{pa^2}{Eh} + C_1 \sin\alpha \sinh\alpha + C_4 \cos\alpha \cosh\alpha &= 0 \\ C_1 \cos\alpha \cosh\alpha - C_4 \sin\alpha \sinh\alpha &= 0 \end{aligned} \qquad (h)$$

where, for the sake of simplicity,

$$\frac{\beta l}{2} = \alpha \qquad (i)$$

From these equations we obtain

$$C_1 = \frac{pa^2}{Eh} \frac{\sin \alpha \sinh \alpha}{\sin^2 \alpha \sinh^2 \alpha + \cos^2 \alpha \cosh^2 \alpha} = \frac{pa^2}{Eh} \frac{2 \sin \alpha \sinh \alpha}{\cos 2\alpha + \cosh 2\alpha}$$
$$C_4 = \frac{pa^2}{Eh} \frac{\cos \alpha \cosh \alpha}{\sin^2 \alpha \sinh^2 \alpha + \cos^2 \alpha \cosh^2 \alpha} = \frac{pa^2}{Eh} \frac{2 \cos \alpha \cosh \alpha}{\cos 2\alpha + \cosh 2\alpha} \qquad (j)$$

Substituting the values (j) and (f) of the constants in expression (e) and observing from expression (275) that

$$\frac{Eh}{a^2} = 4D\beta^4 = \frac{64\alpha^4 D}{l^4} \qquad (k)$$

we obtain

$$w = -\frac{pl^4}{64D\alpha^4}\left(1 - \frac{2 \sin \alpha \sinh \alpha}{\cos 2\alpha + \cosh 2\alpha} \sin \beta x \sinh \beta x - \frac{2 \cos \alpha \cosh \alpha}{\cos 2\alpha + \cosh 2\alpha} \cos \beta x \cosh \beta x\right) \qquad (l)$$

In each particular case, if the dimensions of the shell are known, the quantity α, which is dimensionless, can be calculated by means of notation (i) and Eq. (275). By substituting this value in expression (l) the deflection of the shell at any point can be found.

For the middle of the shell, substituting $x = 0$ in expression (l), we obtain

$$(w)_{x=0} = -\frac{pl^4}{64D\alpha^4}\left(1 - \frac{2 \cos \alpha \cosh \alpha}{\cos 2\alpha + \cosh 2\alpha}\right) \qquad (m)$$

When the shell is long, α becomes large, the second term in the parentheses of expression (m) becomes small, and the deflection approaches the value (d) calculated for the case of free ends. This indicates that in the case of long shells the effect of the end supports upon the deflection at the middle is negligible. Taking another extreme case, *viz.*, the case when α is very small, we can show by expanding the trigonometric and hyperbolic functions in power series that the expression in parentheses in Eq. (m) approaches the value $5\alpha^4/6$ and that the deflection (l) approaches that for a uniformly loaded and simply supported beam of length l and flexural rigidity D.

Differentiating expression (l) twice and multiplying it by D, the bending moment is found as

$$M_x = -D\frac{d^2w}{dx^2} = -\frac{pl^2}{4\alpha^2}\left(\frac{\sin \alpha \sinh \alpha}{\cos 2\alpha + \cosh 2\alpha} \cosh \beta x \cos \beta x - \frac{\cos \alpha \cosh \alpha}{\cos 2\alpha + \cosh 2\alpha} \sin \beta x \sinh \beta x\right) \qquad (n)$$

At the middle of the shell this moment is

$$(M_x)_{x=0} = -\frac{pl^2}{4\alpha^2}\frac{\sin\alpha\sinh\alpha}{\cos 2\alpha + \cosh 2\alpha} \tag{o}$$

It is seen that for large values of α, that is, for long shells, this moment becomes negligibly small and the middle portion is, for all practical purposes, under the action of merely the hoop stresses pa/h.

The case of a cylinder with built-in edges (Fig. 241a) can be treated in a similar manner. Going directly to the final result,[1] we find that the bending moment M_0 acting along the built-in edge is

$$M_0 = \frac{p}{2\beta^2}\frac{\sinh 2\alpha - \sin 2\alpha}{\sinh 2\alpha + \sin 2\alpha} = \frac{p}{2\beta^2}\chi_2(2\alpha) \tag{288}$$

where

$$\chi_2(2\alpha) = \frac{\sinh 2\alpha - \sin 2\alpha}{\sinh 2\alpha + \sin 2\alpha}$$

In the case of long shells, α is large, the factor $\chi_2(2\alpha)$ in expression (288) approaches unity, and the value of the moment approaches that given by the first of the expressions (287). For shorter shells the value of the factor $\chi_2(2\alpha)$ in (288) can be taken from Table 85.

TABLE 85

2α	$\chi_1(2\alpha)$	$\chi_2(2\alpha)$	$\chi_3(2\alpha)$
0.2	5.000	0.0068	0.100
0.4	2.502	0.0268	0.200
0.6	1.674	0.0601	0.300
0.8	1.267	0.1065	0.400
1.0	1.033	0.1670	0.500
1.2	0.890	0.2370	0.596
1.4	0.803	0.3170	0.689
1.6	0.755	0.4080	0.775
1.8	0.735	0.5050	0.855
2.0	0.738	0.6000	0.925
2.5	0.802	0.8220	1.045
3.0	0.893	0.9770	1.090
3.5	0.966	1.0500	1.085
4.0	1.005	1.0580	1.050
4.5	1.017	1.0400	1.027
5.0	1.017	1.0300	1.008

Cylindrical Shell Bent by Forces and Moments Distributed along the Edges. In the preceding section this problem was discussed assuming

[1] Both cases are discussed in detail by I. G. Boobnov in his "Theory of Structure of Ships," vol. 2, p. 368, St. Petersburg, 1913. Also included are numerical tables which simplify the calculations of moments and deflections.

that the shell is long and that each end can be treated independently. In the case of shorter shells both ends must be considered simultaneously by using solution (*e*) with four constants of integration. Proceeding as in the previous cases, the following results can be obtained. For the case of bending by uniformly distributed shearing forces Q_0 (Fig. 242*a*), the deflection and the slope at the ends are

$$
\begin{aligned}
(w)_{x=0,x=l} &= -\frac{2Q_0\beta a^2}{Eh}\frac{\cosh 2\alpha + \cos 2\alpha}{\sinh 2\alpha + \sin 2\alpha} = -\frac{2Q_0\beta a^2}{Eh}\chi_1(2\alpha) \\
\left(\frac{dw}{dx}\right)_{x=0,x=l} &= \pm\frac{2Q_0\beta^2 a^2}{Eh}\frac{\sinh 2\alpha - \sin 2\alpha}{\sinh 2\alpha + \sin 2\alpha} = \pm\frac{2Q_0\beta^2 a^2}{Eh}\chi_2(2\alpha)
\end{aligned}
\tag{289}
$$

In the case of bending by the moments M_0 (Fig. 242*b*), we obtain

$$
\begin{aligned}
(w)_{x=0,x=l} &= -\frac{2M_0\beta^2 a^2}{Eh}\frac{\sinh 2\alpha - \sin 2\alpha}{\sinh 2\alpha + \sin 2\alpha} = -\frac{2M_0\beta^2 a^2}{Eh}\chi_2(2\alpha) \\
\left(\frac{dw}{dx}\right)_{x=0,x=l} &= \pm\frac{4M_0\beta^3 a^2}{Eh}\frac{\cosh 2\alpha - \cos 2\alpha}{\sinh 2\alpha + \sin 2\alpha} = \pm\frac{4M_0\beta^3 a^2}{Eh}\chi_3(2\alpha)
\end{aligned}
\tag{290}
$$

In the case of long shells, the factors χ_1, χ_2, and χ_3 in expressions (289) and (290) are close to unity, and the results coincide with those given by

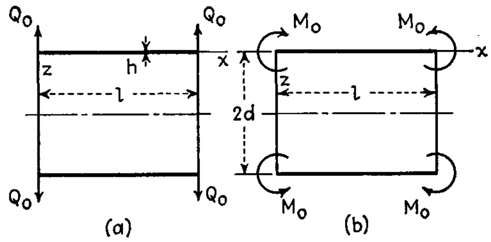

FIG. 242

expressions (279) and (280). To simplify the calculations for shorter shells, the values of functions χ_1, χ_2, and χ_3 are given in Table 85.

Using solutions (289) and (290), the stresses in a long pipe reinforced by equidistant rings (Fig. 243) and submitted to the action of uniform internal pressure p can be readily discussed.

Assume first that there are no rings. Then, under the action of internal pressure, hoop stresses $\sigma_t = pa/h$ will be produced, and the radius of the pipe will increase by the amount

$$\delta = \frac{pa^2}{Eh}$$

Now, taking the rings into consideration and assuming that they are absolutely rigid, we conclude that reactive forces will be produced between each ring and the pipe. The magnitude of the forces per unit length of the circumference of the tube will be denoted by P. The magnitude of P will now be determined from the condition that the forces P produce a deflection of the pipe under the ring equal to the expansion δ created by the internal pressure p. In calculating this deflection we observe that a portion of the tube between two adjacent rings may be considered as the shell shown in Fig. 242a and b. In this case $Q_0 = -\frac{1}{2}P$, and the magnitude of the bending moment M_0 under a ring is determined from the condition that $dw/dx = 0$ at that point. Hence from Eqs. (289) and (290) we find

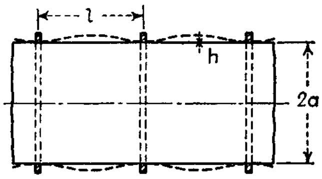

FIG. 243

$$-\frac{P\beta^2a^2}{Eh}\chi_2(2\alpha) + \frac{4M_0\beta^3a^2}{Eh}\chi_3(2\alpha) = 0$$

from which

$$M_0 = \frac{P\chi_2(2\alpha)}{4\beta\chi_3(2\alpha)} \tag{p}$$

If the distance l between the rings is large,[1] the quantity

$$2\alpha = \beta l = \frac{l}{\sqrt{ah}}\sqrt[4]{3(1-\nu^2)}$$

is also large, the functions $\chi_2(2\alpha)$ and $\chi_3(2\alpha)$ approach unity, and the moment M_0 approaches the value (286). For calculating the force P entering in Eq. (p) the expressions for deflections as given in Eqs. (289) and (290) must be used. These expressions give

$$\frac{P\beta a^2}{Eh}\chi_1(2\alpha) - \frac{P\beta a^2}{2Eh}\frac{\chi_2^2(2\alpha)}{\chi_3(2\alpha)} = \delta = \frac{pa^2}{Eh}$$

or

$$P\beta\left[\chi_1(2\alpha) - \frac{1}{2}\frac{\chi_2^2(2\alpha)}{\chi_3(2\alpha)}\right] = \frac{\delta Eh}{a^2} = p \tag{291}$$

For large values of 2α this reduces to

$$\frac{P\beta a^2}{2Eh} = \delta$$

which coincides with Eq. (285). When 2α is not large, the value of the reactive forces P is calculated from Eq. (291) by using Table 85. Solving Eq. (291) for P and substituting its expression in expression (p), we find

$$M_0 = \frac{p}{2\beta^2}\chi_2(2\alpha) \tag{292}$$

This coincides with expression (288) previously obtained for a shell with built-in edges.

To take into account the extension of rings we observe that the reactive

[1] For $\nu = 0.3$, $2\alpha = 1.285l/\sqrt{ah}$.

forces P produce in the ring a tensile force Pa and that the corresponding increase of the inner radius of the ring is[1]

$$\delta_1 = \frac{Pa^2}{AE}$$

where A is the cross-sectional area of the ring. To take this extension into account we substitute $\delta - \delta_1$ for δ in Eq. (291) and obtain

$$P\beta\left[\chi_1(2\alpha) - \frac{1}{2}\frac{\chi_2^2(2\alpha)}{\chi_3(2\alpha)}\right] = p - \frac{Ph}{A} \tag{293}$$

From this equation, P can be readily obtained by using Table 85, and the moment found by substituting $p - (Ph/A)$ for p in Eq. (292).

If the pressure p acts not only on the cylindrical shell but also on the ends, longitudinal forces

$$N_x = \frac{pa}{2}$$

are produced in the shell. The extension of the radius of the cylinder is then

$$\delta' = \frac{pa^2}{Eh}\left(1 - \frac{1}{2}\nu\right)$$

and the quantity $p(1 - \frac{1}{2}\nu)$ must be substituted for p in Eqs. (292) and (293).

Equations (293) and (291) can also be used in the case of external uniform pressure provided the compressive stresses in the ring and in the shell are far enough from the critical stresses at which buckling may occur.[2] This case is of practical importance in the design of submarines and has been discussed by several authors.[3]

116. Pressure Vessels. The method illustrated by the examples of the preceding article can also be applied in the analysis of stresses in cylindrical vessels submitted to the action of internal pressure.[4] In discussing the "membrane theory" it was repeatedly indicated that this theory fails to represent the true stresses in those portions of a shell close to the

[1] It is assumed that the cross-sectional dimensions of the ring are small in comparison with the radius a.

[2] Buckling of rings and cylindrical shells is discussed in S. Timoshenko, "Theory of Elastic Stability," 1936.

[3] See paper by K. von Sanden and K. Günther, "Werft und Reederei," vol. 1, 1920, pp. 163–168, 189–198, 216–221, and vol. 2, 1921, pp. 505–510.

[4] See also M. Esslinger, "Statische Berechnung von Kesselböden," Berlin, 1952; G. Salet and J. Barthelemy, *Bull. Assoc. Tech. Maritime Aeronaut.*, vol. 44, p. 505, 1945; J. L. Maulbetsch and M. Hetényi, *ASCE Design Data*, no. 1, 1944, and F. Schultz-Grunow, *Ingr.-Arch.*, vol. 4, p. 545, 1933; N. L. Svensson, *J. Appl. Mechanics*, vol. 25, p. 89, 1958.

edges, since the edge conditions usually cannot be completely satisfied by considering only membrane stresses. A similar condition in which the membrane theory is inadequate is found in cylindrical pressure vessels at the joints between the cylindrical portion and the ends of the vessel. At these joints the membrane stresses are usually accompanied by local bending stresses which are distributed symmetrically with respect to the axis of the cylinder. These local stresses can be calculated by using solution (278) of Art. 114.

Let us begin with the simple case of a cylindrical vessel with hemispherical ends (Fig. 244).[1] At a sufficient distance from the joints mn

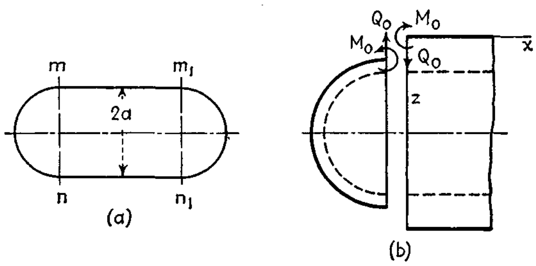

FIG. 244

and m_1n_1 the membrane theory is accurate enough and gives for the cylindrical portion of radius a

$$N_x = \frac{pa}{2} \qquad N_t = pa \tag{a}$$

where p denotes the internal pressure.

For the spherical ends this theory gives a uniform tensile force

$$N = \frac{pa}{2} \tag{b}$$

The extension of the radius of the cylindrical shell under the action of the forces (a) is

$$\delta_1 = \frac{pa^2}{Eh}\left(1 - \frac{\nu}{2}\right) \tag{c}$$

and the extension of the radius of the spherical ends is

$$\delta_2 = \frac{pa^2}{2Eh}(1 - \nu) \tag{d}$$

Comparing expressions (c) and (d), it can be concluded that if we consider only membrane stresses we obtain a discontinuity at the joints as represented in Fig. 244b. This indicates that at the joint there must act

[1] This case was discussed by E. Meissner, *Schweiz. Bauztg.*, vol. 86, p. 1, 1925.

shearing forces Q_0 and bending moments M_0 uniformly distributed along the circumference and of such magnitudes as to eliminate this discontinuity. The stresses produced by these forces are sometimes called *discontinuity stresses.*

In calculating the quantities Q_0 and M_0 we assume that the bending is of a local character so that solution (278) can be applied with sufficient accuracy in discussing the bending of the cylindrical portion. The investigation of the bending of the spherical ends represents a more complicated problem which will be fully discussed in Chap. 16. Here we obtain an approximate solution of the problem by assuming that the bending is of importance only in the zone of the spherical shell close to the joint and that this zone can be treated as a portion of a long cylindrical shell[1] of radius a. If the thickness of the spherical and the cylindrical portion of the vessel is the same, the forces Q_0 produce equal rotations of the edges of both portions at the joint (Fig. 244*b*). This indicates that M_0 vanishes and that Q_0 alone is sufficient to eliminate the discontinuity. The magnitude of Q_0 is now determined from the condition that the sum of the numerical values of the deflections of the edges of the two parts must be equal to the difference $\delta_1 - \delta_2$ of the radial expansions furnished by the membrane theory. Using Eq. (279) for the deflections, we obtain

$$\frac{Q_0}{\beta^3 D} = \delta_1 - \delta_2 = \frac{pa^2}{2Eh}$$

from which, by using notation (275),

$$Q_0 = \frac{pa^2\beta^3 D}{2Eh} = \frac{p}{8\beta} \tag{e}$$

Having obtained this value of the force Q_0, the deflection and the bending moment M_x can be calculated at any point by using formulas (282), which give[2]

$$w = \frac{Q_0}{2\beta^3 D}\,\theta(\beta x)$$

$$M_x = -D\frac{d^2w}{dx^2} = -\frac{Q_0}{\beta}\,\zeta(\beta x)$$

Substituting expression (*e*) for Q_0 and expression (275) for β in the formula for M_x, we obtain

$$M_x = -\frac{ahp}{8\sqrt{3(1-\nu^2)}}\,\zeta(\beta x) \tag{f}$$

[1] E. Meissner, in the above-mentioned paper, showed that the error in the magnitude of the bending stresses as calculated from such an approximate solution is small for thin hemispherical shells and is smaller than 1 per cent if $a/h > 30$.

[2] Note that the direction of Q_0 in Fig. 244 is opposite to the direction in Fig. 236.

This moment attains its numerical maximum at the distance $x = \pi/4\beta$, at which point the derivative of the moment is zero, as can be seen from the fourth of the equations (282).

Combining the maximum bending stress produced by M_x with the membrane stress, we find

$$(\sigma_x)_{\max} = \frac{ap}{2h} + \frac{3}{4}\frac{ap}{h\sqrt{3(1-\nu^2)}}\zeta\left(\frac{\pi}{4}\right) = 1.293\frac{ap}{2h} \qquad (g)$$

This stress which acts at the outer surface of the cylindrical shell is about 30 per cent larger than the membrane stress acting in the axial direction. In calculating stresses in the circumferential direction in addition to the membrane stress pa/h, the hoop stress caused by the deflection w as well as the bending stress produced by the moment $M_\varphi = \nu M_x$ must be considered. In this way we obtain at the outer surface of the cylindrical shell

$$\sigma_t = \frac{ap}{h} - \frac{Ew}{a} - \frac{6\nu}{h^2}M_x = \frac{ap}{h}\left[1 - \frac{1}{4}\theta(\beta x) + \frac{3\nu}{4\sqrt{3(1-\nu^2)}}\zeta(\beta x)\right]$$

Taking $\nu = 0.3$ and using Table 84, we find

$$(\sigma_t)_{\max} = 1.032\frac{ap}{h} \qquad \text{at } \beta x = 1.85 \qquad (h)$$

Since the membrane stress is smaller in the ends than in the cylinder sides, the maximum stress in the spherical ends is always smaller than the calculated stress (h). Thus the latter stress is the determining factor in the design of the vessel.

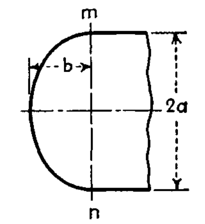

FIG. 245

The same method of calculating discontinuity stresses can be applied in the case of ends having the form of an ellipsoid of revolution. The membrane stresses in this case are obtained from expressions (263) and (264) (see page 440). At the joint mn which represents the equator of the ellipsoid (Fig. 245), the stresses in the direction of the meridian and in the equatorial direction are, respectively,

$$\sigma_\varphi = \frac{pa}{2h} \qquad \sigma_\theta = \frac{pa}{h}\left(1 - \frac{a^2}{2b^2}\right) \qquad (i)$$

The extension of the radius of the equator is

$$\delta_2' = \frac{a}{E}(\sigma_\theta - \nu\sigma_\varphi) = \frac{pa^2}{Eh}\left(1 - \frac{a^2}{2b^2} - \frac{\nu}{2}\right)$$

Substituting this quantity for δ_2 in the previous calculation of the shearing force Q_0, we find

$$\delta_1 - \delta_2' = \frac{pa^2}{Eh}\frac{a^2}{2b^2}$$

and, instead of Eq. (*e*), we obtain

$$Q_0 = \frac{p}{8\beta}\frac{a^2}{b^2}$$

It is seen that the shearing force Q_0 in the case of ellipsoidal ends is larger than in the case of hemispherical ends in the ratio a^2/b^2. The discontinuity stresses will evidently increase in the same proportion. For example, taking $a/b = 2$, we obtain, from expressions (*g*) and (*h*),

$$(\sigma_x)_{\max} = \frac{ap}{2h} + \frac{3ap}{h\sqrt{3(1-\nu^2)}}\zeta\left(\frac{\pi}{4}\right) = 2.172\frac{ap}{2h}$$

$$(\sigma_t)_{\max} = 1.128\frac{ap}{h}$$

Again, $(\sigma_t)_{\max}$ is the largest stress and is consequently the determining factor in design.[1]

117. Cylindrical Tanks with Uniform Wall Thickness. If a tank is submitted to the action of a liquid pressure, as shown in Fig. 246, the stresses in the wall can be analyzed by using Eq. (276). Substituting in this equation

$$Z = -\gamma(d - x) \qquad (a)$$

where γ is the weight per unit volume of the liquid, we obtain

$$\frac{d^4w}{dx^4} + 4\beta^4 w = -\frac{\gamma(d-x)}{D} \qquad (b)$$

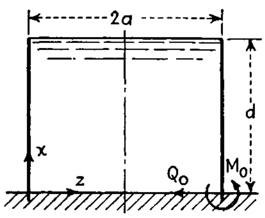

FIG. 246

A particular solution of this equation is

$$w_1 = -\frac{\gamma(d-x)}{4\beta^4 D} = -\frac{\gamma(d-x)a^2}{Eh} \qquad (c)$$

This expression represents the radial expansion of a cylindrical shell with free edges under the action of hoop stresses. Substituting expression (*c*) in place of $f(x)$ in expression (277), we obtain for the complete solution of Eq. (*b*)

$$w = e^{\beta x}(C_1\cos\beta x + C_2\sin\beta x) + e^{-\beta x}(C_3\cos\beta x + C_4\sin\beta x) - \frac{\gamma(d-x)a^2}{Eh}$$

In most practical cases the wall thickness h is small in comparison with both the radius a and the depth d of the tank, and we may consider the shell as infinitely long. The constants C_1 and C_2 are then equal to zero,

[1] More detail regarding stresses in boilers with ellipsoidal ends can be found in the book by Höhn, "Über die Festigkeit der gewölbten Böden und der Zylinderschale," Zürich, 1927. Also included are the results of experimental investigations of discontinuity stresses which are in a good agreement with the approximate solution. See also Schultz-Grunow, *loc. cit.*

and we obtain

$$w = e^{-\beta x}(C_3 \cos \beta x + C_4 \sin \beta x) - \frac{\gamma(d - x)a^2}{Eh} \qquad (d)$$

The constants C_3 and C_4 can now be obtained from the conditions at the bottom of the tank. Assuming that the lower edge of the wall is built into an absolutely rigid foundation, the boundary conditions are

$$(w)_{x=0} = C_3 - \frac{\gamma a^2 d}{Eh} = 0$$

$$\left(\frac{dw}{dx}\right)_{x=0} = \left[-\beta C_3 e^{-\beta x}(\cos \beta x + \sin \beta x) + \beta C_4 e^{-\beta x}(\cos \beta x - \sin \beta x) + \frac{\gamma a^2}{Eh}\right]_{x=0} = \beta(C_4 - C_3) + \frac{\gamma a^2}{Eh} = 0$$

From these equations we obtain

$$C_3 = \frac{\gamma a^2 d}{Eh} \qquad C_4 = \frac{\gamma a^2}{Eh}\left(d - \frac{1}{\beta}\right)$$

Expression (d) then becomes

$$w = -\frac{\gamma a^2}{Eh}\left\{d - x - e^{-\beta x}\left[d \cos \beta x + \left(d - \frac{1}{\beta}\right) \sin \beta x\right]\right\}$$

from which, by using the notation of Eqs. (281), we obtain

$$w = -\frac{\gamma a^2 d}{Eh}\left[1 - \frac{x}{d} - \theta(\beta x) - \left(1 - \frac{1}{\beta d}\right)\zeta(\beta x)\right] \qquad (e)$$

From this expression the deflection at any point can be readily calculated by the use of Table 84. The force N_φ in the circumferential direction is then

$$N_\varphi = -\frac{Ehw}{a} = \gamma a d\left[1 - \frac{x}{d} - \theta(\beta x) - \left(1 - \frac{1}{\beta d}\right)\zeta(\beta x)\right] \qquad (f)$$

From the second derivative of expression (e) we obtain the bending moment

$$M_x = -D\frac{d^2w}{dx^2} = \frac{2\beta^2\gamma a^2 D d}{Eh}\left[-\zeta(\beta x) + \left(1 - \frac{1}{\beta d}\right)\theta(\beta x)\right]$$
$$= \frac{\gamma a d h}{\sqrt{12(1 - \nu^2)}}\left[-\zeta(\beta x) + \left(1 - \frac{1}{\beta d}\right)\theta(\beta x)\right] \qquad (g)$$

Having expressions (f) and (g), the maximum stress at any point can readily be calculated in each particular case. The bending moment has its maximum value at the bottom, where it is equal to

$$(M_x)_{x=0} = M_0 = \left(1 - \frac{1}{\beta d}\right)\frac{\gamma a d h}{\sqrt{12(1 - \nu^2)}} \qquad (h)$$

The same result can be obtained by using solutions (279) and (280) (pages 469, 470). Assuming that the lower edge of the shell is entirely free, we obtain from expression (*c*)

$$(w_1)_{x=0} = -\frac{\gamma a^2 d}{Eh} \qquad \left(\frac{dw_1}{dx}\right)_{x=0} = \frac{\gamma a^2}{Eh} \tag{i}$$

To eliminate this displacement and rotation of the edge and thus satisfy the edge conditions at the bottom of the tank, a shearing force Q_0 and bending moment M_0 must be applied as indicated in Fig. 246. The magnitude of each of these quantities is obtained by equating expressions (279) and (280) to expressions (*i*) taken with reversed signs. This gives

$$-\frac{1}{2\beta^3 D}(\beta M_0 + Q_0) = +\frac{\gamma a^2 d}{Eh}$$
$$\frac{1}{2\beta^2 D}(2\beta M_0 + Q_0) = -\frac{\gamma a^2}{Eh}$$

From these equations we again obtain expression (*h*) for M_0, whereas for the shearing force we find[1]

$$Q_0 = -\frac{\gamma a d h}{\sqrt{12(1-\nu^2)}}\left(2\beta - \frac{1}{d}\right) \tag{j}$$

Taking, as an example, $a = 30$ ft, $d = 26$ ft, $h = 14$ in., $\gamma = 0.03613$ lb per in.^{3}, and $\nu = 0.25$, we find $\beta = 0.01824$ in.$^{-1}$ and $\beta d = 5.691$. For such a value of βd our assumption that the shell is infinitely long results in an accurate value for the moment and the shearing force, and we obtain from expressions (*h*) and (*j*)

$M_0 = 13{,}960$ in.-lb per in. $\qquad Q_0 = -563.6$ lb per in.

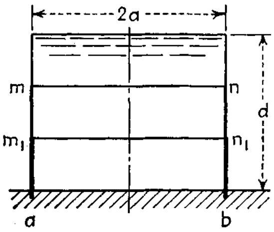

FIG. 247

In the construction of steel tanks, metallic sheets of several different thicknesses are very often used as shown in Fig. 247. Applying the particular solution (*c*) to each portion of uniform thickness, we find that the differences in thickness give rise to discontinuities in the displacement w_1 along the joints mn and m_1n_1. These discontinuities, together with the displacements at the bottom ab, can be removed by applying moments and shearing forces. Assuming that the vertical dimension of each portion is sufficiently large to justify the application of the formulas for an infinitely large shell, we calculate the discontinuity moments and shearing forces as before by using Eqs. (279) and (280) and applying at each joint the two conditions that the adjacent portions of the shell have equal deflections and a common tangent. If the use of formulas (279) and (280) derived for an infinitely long shell cannot be justified, the general solution containing four constants of integration must be applied to each portion of the tank. The determination of the constants under such conditions becomes much more complicated, since the fact that each joint cannot be treated

[1] The negative sign indicates that Q_0 has the direction shown in Fig. 246 which is opposite to the direction used in Fig. 236 when deriving expressions (279) and (280).