CHAPTER 5

SIMPLY SUPPORTED RECTANGULAR PLATES

27. Simply Supported Rectangular Plates under Sinusoidal Load. Taking the coordinate axes as shown in Fig. 59, we assume that the load distributed over the surface of the plate is given by the expression

$$q = q_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \qquad (a)$$

in which q_0 represents the intensity of the load at the center of the plate. The differential equation (103) for the deflection surface in this case becomes

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q_0}{D} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \qquad (b)$$

The boundary conditions for simply supported edges are

$$w = 0 \qquad M_x = 0 \qquad \text{for } x = 0 \text{ and } x = a$$
$$w = 0 \qquad M_y = 0 \qquad \text{for } y = 0 \text{ and } y = b$$

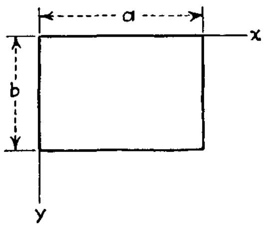

FIG. 59

Using expression (101) for bending moments and observing that, since $w = 0$ at the edges, $\partial^2 w/\partial x^2 = 0$ and $\partial^2 w/\partial y^2 = 0$ for the edges parallel to the x and y axes, respectively, we can represent the boundary conditions in the following form:

$$\begin{aligned} &(1)\ w = 0 \qquad (2)\ \frac{\partial^2 w}{\partial x^2} = 0 \qquad \text{for } x = 0 \text{ and } x = a \\ &(3)\ w = 0 \qquad (4)\ \frac{\partial^2 w}{\partial y^2} = 0 \qquad \text{for } y = 0 \text{ and } y = b \end{aligned} \qquad (c)$$

It may be seen that all boundary conditions are satisfied if we take for deflections the expression

$$w = C \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \qquad (d)$$

in which the constant C must be chosen so as to satisfy Eq. (b). Substituting expression (d) into Eq. (b), we find

$$\pi^4 \left(\frac{1}{a^2} + \frac{1}{b^2} \right)^2 C = \frac{q_0}{D}$$

and we conclude that the deflection surface satisfying Eq. (*b*) and boundary conditions (*c*) is

$$w = \frac{q_0}{\pi^4 D \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \tag{e}$$

Having this expression and using Eqs. (101) and (102), we find

$$\begin{aligned} M_x &= \frac{q_0}{\pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2} \left(\frac{1}{a^2} + \frac{\nu}{b^2}\right) \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \\ M_y &= \frac{q_0}{\pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2} \left(\frac{\nu}{a^2} + \frac{1}{b^2}\right) \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \\ M_{xy} &= \frac{q_0(1-\nu)}{\pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2 ab} \cos \frac{\pi x}{a} \cos \frac{\pi y}{b} \end{aligned} \tag{f}$$

It is seen that the maximum deflection and the maximum bending moments are at the center of the plate. Substituting $x = a/2$, $y = b/2$ in Eqs. (*e*) and (*f*), we obtain

$$w_{\max} = \frac{q_0}{\pi^4 D \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2} \tag{124}$$

$$\begin{aligned} (M_x)_{\max} &= \frac{q_0}{\pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2} \left(\frac{1}{a^2} + \frac{\nu}{b^2}\right) \\ (M_y)_{\max} &= \frac{q_0}{\pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2} \left(\frac{\nu}{a^2} + \frac{1}{b^2}\right) \end{aligned} \tag{125}$$

In the particular case of a square plate, $a = b$, and the foregoing formulas become

$$w_{\max} = \frac{q_0 a^4}{4\pi^4 D} \qquad (M_x)_{\max} = (M_y)_{\max} = \frac{(1+\nu) q_0 a^2}{4\pi^2} \tag{126}$$

We use Eqs. (106) and (107) to calculate the shearing forces and obtain

$$\begin{aligned} Q_x &= \frac{q_0}{\pi a \left(\frac{1}{a^2} + \frac{1}{b^2}\right)} \cos \frac{\pi x}{a} \sin \frac{\pi y}{b} \\ Q_y &= \frac{q_0}{\pi b \left(\frac{1}{a^2} + \frac{1}{b^2}\right)} \sin \frac{\pi x}{a} \cos \frac{\pi y}{b} \end{aligned} \tag{g}$$

To find the reactive forces at the supported edges of the plate we proceed as was explained in Art. 22. For the edge $x = a$ we find

$$V_x = \left(Q_x - \frac{\partial M_{xy}}{\partial y}\right)_{x=a} = -\frac{q_0}{\pi a\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2}\left(\frac{1}{a^2} + \frac{2-\nu}{b^2}\right)\sin\frac{\pi y}{b} \qquad (h)$$

In the same manner, for the edge $y = b$,

$$V_y = \left(Q_y - \frac{\partial M_{xy}}{\partial x}\right)_{y=b} = -\frac{q_0}{\pi b\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2}\left(\frac{1}{b^2} + \frac{2-\nu}{a^2}\right)\sin\frac{\pi x}{a} \qquad (i)$$

Hence the pressure distribution follows a sinusoidal law. The minus sign indicates that the reactions on the plate act upward. From symmetry it may be concluded that formulas (h) and (i) also represent pressure distributions along the sides $x = 0$ and $y = 0$, respectively. The resultant of distributed pressures is

$$\frac{2q_0}{\pi\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2}\left[\frac{1}{a}\left(\frac{1}{a^2} + \frac{2-\nu}{b^2}\right)\int_0^b \sin\frac{\pi y}{b}\,dy + \frac{1}{b}\left(\frac{1}{b^2} + \frac{2-\nu}{a^2}\right)\int_0^a \sin\frac{\pi x}{a}\,dx\right] = \frac{4q_0ab}{\pi^2} + \frac{8q_0(1-\nu)}{\pi^2 ab\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2} \qquad (j)$$

Observing that

$$\frac{4q_0ab}{\pi^2} = \int_0^a\int_0^b q_0 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\,dx\,dy \qquad (k)$$

it can be concluded that the sum of the distributed reactions is larger than the total load on the plate given by expression (k). This result can be easily explained if we note that, proceeding as described in Art. 22, we obtain not only the distributed reactions but also reactions concentrated at the corners of the plate. These concentrated reactions are equal, from symmetry; and their magnitude, as may be seen from Fig. 51, is

$$R = 2(M_{xy})_{x=a,y=b} = \frac{2q_0(1-\nu)}{\pi^2 ab\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2} \qquad (l)$$

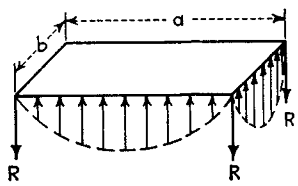

FIG. 60

The positive sign indicates that the reactions act downward. Their sum is exactly equal to the second term in expression (j). The distributed and the concentrated reactions which act on the plate and keep the load, defined by Eq. (a), in equilibrium are shown graphically in Fig. 60. It may be seen that the corners of the plate have a tendency to rise up under the action of the applied load and that the concentrated forces R must be applied to prevent this.

The maximum bending stress is at the center of the plate. Assuming that $a > b$, we find that at the center $M_y > M_x$. Hence the maximum bending stress is

$$(\sigma_y)_{\max} = \frac{6(M_y)_{\max}}{h^2} = \frac{6q_0}{\pi^2 h^2 \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2} \left(\frac{\nu}{a^2} + \frac{1}{b^2}\right)$$

The maximum shearing stress will be at the middle of the longer sides of the plate. Observing that the total transverse force $V_y = Q_y - \frac{\partial M_{xy}}{\partial x}$ is distributed along the thickness of the plate according to the parabolic law and using Eq. (i), we obtain

$$(\tau_{yz})_{\max} = \frac{3q_0}{2\pi b h \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2} \left(\frac{1}{b^2} + \frac{2 - \nu}{a^2}\right)$$

If the sinusoidal load distribution is given by the equation

$$q = q_0 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{m}$$

where m and n are integer numbers, we can proceed as before, and we shall obtain for the deflection surface the following expression:

$$w = \frac{q_0}{\pi^4 D \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{127}$$

from which the expressions for bending and twisting moments can be readily obtained by differentiation.

28. Navier Solution for Simply Supported Rectangular Plates. The solution of the preceding article can be used in calculating deflections produced in a simply supported rectangular plate by any kind of loading given by the equation

$$q = f(x,y) \tag{a}$$

For this purpose we represent the function $f(x,y)$ in the form of a double trigonometric series:[1]

$$f(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{128}$$

[1] The first solution of the problem of bending of simply supported rectangular plates and the use for this purpose of double trigonometric series are due to Navier, who

To calculate any particular coefficient $a_{m'n'}$ of this series we multiply both sides of Eq. (128) by $\sin (n'\pi y/b)\, dy$ and integrate from 0 to b. Observing that

$$\int_0^b \sin \frac{n\pi y}{b} \sin \frac{n'\pi y}{b}\, dy = 0 \qquad \text{when } n \neq n'$$

$$\int_0^b \sin \frac{n\pi y}{b} \sin \frac{n'\pi y}{b}\, dy = \frac{b}{2} \qquad \text{when } n = n'$$

we find in this way

$$\int_0^b f(x,y) \sin \frac{n'\pi y}{b}\, dy = \frac{b}{2} \sum_{m=1}^{\infty} a_{mn'} \sin \frac{m\pi x}{a} \tag{b}$$

Multiplying both sides of Eq. (b) by $\sin (m'\pi x/a)\, dx$ and integrating from 0 to a, we obtain

$$\int_0^a \int_0^b f(x,y) \sin \frac{m'\pi x}{a} \sin \frac{n'\pi y}{b}\, dx\, dy = \frac{ab}{4} a_{m'n'}$$

from which

$$a_{m'n'} = \frac{4}{ab} \int_0^a \int_0^b f(x,y) \sin \frac{m'\pi x}{a} \sin \frac{n'\pi y}{b}\, dx\, dy \tag{129}$$

Performing the integration indicated in expression (129) for a given load distribution, *i.e.*, for a given $f(x,y)$, we find the coefficients of series (128) and represent in this way the given load as a sum of partial sinusoidal loadings. The deflection produced by each partial loading was discussed in the preceding article, and the total deflection will be obtained by summation of such terms as are given by Eq. (127). Hence we find

$$w = \frac{1}{\pi^4 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{mn}}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{130}$$

Take the case of a load uniformly distributed over the entire surface of the plate as an example of the application of the general solution (130). In such a case

$$f(x,y) = q_0$$

where q_0 is the intensity of the uniformly distributed load. From formula (129) we obtain

$$a_{mn} = \frac{4q_0}{ab} \int_0^a \int_0^b \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy = \frac{16q_0}{\pi^2 mn} \tag{c}$$

presented a paper on this subject to the French Academy in 1820. The abstract of the paper was published in *Bull. soc. phil.-math.*, Paris, 1823. The manuscript is in the library of l'École des Ponts et Chaussées.

where m and n are odd integers. If m or n or both of them are even numbers, $a_{mn} = 0$. Substituting in Eq. (130), we find

$$w = \frac{16q_0}{\pi^6 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}}{mn \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2} \tag{131}$$

where $m = 1, 3, 5, \ldots$ and $n = 1, 3, 5, \ldots$.

In the case of a uniform load we have a deflection surface symmetrical with respect to the axes $x = a/2$, $y = b/2$; and quite naturally all terms with even numbers for m or n in series (131) vanish, since they are unsymmetrical with respect to the above-mentioned axes. The maximum deflection of the plate is at its center and is found by substituting $x = a/2$, $y = b/2$ in formula (131), giving

$$w_{\max} = \frac{16q_0}{\pi^6 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{\frac{m+n}{2}-1}}{mn \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2} \tag{132}$$

This is a rapidly converging series, and a satisfactory approximation is obtained by taking only the first term of the series, which, for example, in the case of a square plate gives

$$w_{\max} = \frac{4q_0 a^4}{\pi^6 D} = 0.00416 \frac{q_0 a^4}{D}$$

or, by substituting expression (3) for D and assuming $\nu = 0.3$,

$$w_{\max} = 0.0454 \frac{q_0 a^4}{Eh^3}$$

This result is about $2\frac{1}{2}$ per cent in error (see Table 8).

From expression (132) it may be seen that the deflections of two plates that have the same thickness and the same value of the ratio a/b increase as the fourth power of the length of the sides.

The expressions for bending and twisting moments can be obtained from the general solution (131) by using Eqs. (101) and (102). The series obtained in this way are not so rapidly convergent as series (131), and in the further discussion (see Art. 30) another form of solution will be given, more suitable for numerical calculations. Since the moments are expressed by the second derivatives of series (131), their maximum values, if we keep q_0 and D the same, are proportional to the square of linear dimensions. Since the total load on the plate, equal to $q_0 ab$, is also proportional to the square of the linear dimensions, we conclude that, for two plates of equal thickness and of the same value of the ratio a/b, the maximum bending moments and hence the maximum stresses are equal if the total loads on the two plates are equal.[1]

29. Further Applications of the Navier Solution. From the discussion in the preceding article it is seen that the deflection of a simply supported rectangular plate (Fig. 59) can always be represented in the form of a double trigonometric series (130), the coefficients a_{mn} being given by Eq. (129).

Let us apply this result in the case of a single load P uniformly distributed over the area of the rectangle shown in Fig. 61. By virtue of Eq. (129) we have

$$a_{mn} = \frac{4P}{abuv} \int_{\xi-u/2}^{\xi+u/2} \int_{\eta-v/2}^{\eta+v/2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy$$

or

$$a_{mn} = \frac{16P}{\pi^2 mnuv} \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b} \sin\frac{m\pi u}{2a} \sin\frac{n\pi v}{2b} \qquad (a)$$

If, in particular, $\xi = a/2$, $\eta = b/2$, $u = a$, and $v = b$, Eq. (a) yields the expression (c) obtained in Art. 28 for the uniformly loaded plate.

Another case of practical interest is a single load concentrated at any given point $x = \xi$, $y = \eta$ of the plate. Using Eq. (a) and letting u and v tend to zero we arrive at the expression

$$a_{mn} = \frac{4P}{ab} \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b} \qquad (b)$$

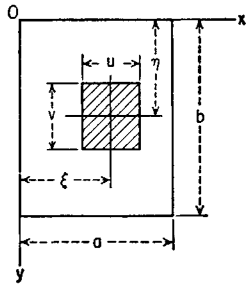

FIG. 61

and, by Eq. (130), at the deflection

$$w = \frac{4P}{\pi^4 abD} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b}}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad (133)$$

The series converges rapidly, and we can obtain the deflection at any point of the plate with sufficient accuracy by taking only the first few terms of the series. Let us, for example, calculate the deflection at the middle when the load is applied at the middle as well. Then we have $\xi = x = a/2$, $\eta = y = b/2$, and the series (133) yields

$$w_{\max} = \frac{4P}{\pi^4 abD} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \qquad (c)$$

[1] This conclusion was established by Mariotte in the paper "Traité du mouvement des eaux," published in 1686. See Mariotte's scientific papers, new ed., vol. 2, p. 467, 1740.