CHAPTER 4

SMALL DEFLECTIONS OF LATERALLY LOADED PLATES

21. The Differential Equation of the Deflection Surface. We assume that the load acting on a plate is normal to its surface and that the deflections are small in comparison with the thickness of the plate (see Art. 13). At the boundary we assume that the edges of the plate are free to move in the plane of the plate; thus the reactive forces at the edges are normal to the plate. With these assumptions we can neglect any strain in the middle plane of the plate during bending. Taking, as

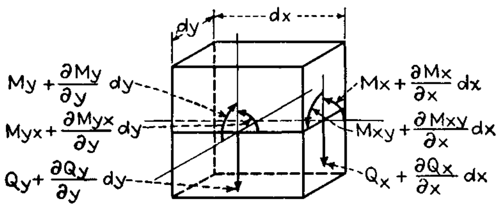

FIG. 47

before (see Art. 10), the coordinate axes x and y in the middle plane of the plate and the z axis perpendicular to that plane, let us consider an element cut out of the plate by two pairs of planes parallel to the xz and yz planes, as shown in Fig. 47. In addition to the bending moments M_x and M_y and the twisting moments M_{xy} which were considered in the pure bending of a plate (see Art. 10), there are vertical shearing forces[1] acting on the sides of the element. The magnitudes of these shearing forces per unit length parallel to the y and x axes we denote by Q_x and Q_y, respectively, so that

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz}\, dz \qquad Q_y = \int_{-h/2}^{h/2} \tau_{yz}\, dz \tag{a}$$

Since the moments and the shearing forces are functions of the coordinates x and y, we must, in discussing the conditions of equilibrium of the element, take into consideration the small changes of these quantities when the coordinates x and y change by the small quantities dx and dy.

[1] There will be no horizontal shearing forces and no forces normal to the sides of the element, since the strain of the middle plane of the plate is assumed negligible.

The middle plane of the element is represented in Fig. 48*a* and *b*, and the directions in which the moments and forces are taken as positive are indicated.

We must also consider the load distributed over the upper surface of the plate. The intensity of this load we denote by q, so that the load acting on the element[1] is $q\,dx\,dy$.

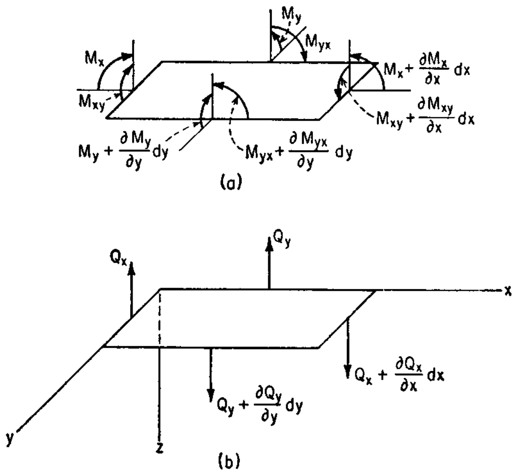

FIG. 48

Projecting all the forces acting on the element onto the z axis we obtain the following equation of equilibrium:

$$\frac{\partial Q_x}{\partial x}\,dx\,dy + \frac{\partial Q_y}{\partial y}\,dy\,dx + q\,dx\,dy = 0$$

from which

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \tag{99}$$

Taking moments of all the forces acting on the element with respect to the x axis, we obtain the equation of equilibrium

$$\frac{\partial M_{xy}}{\partial x}\,dx\,dy - \frac{\partial M_y}{\partial y}\,dy\,dx + Q_y\,dx\,dy = 0 \tag{b}$$

[1] Since the stress component σ_z is neglected, we actually are not able to apply the load on the upper or on the lower surface of the plate. Thus, every transverse single load considered in the thin-plate theory is merely a discontinuity in the magnitude of the shearing forces, which vary according to the parabolic law through the thickness of the plate. Likewise, the weight of the plate can be included in the load q without affecting the accuracy of the result. If the effect of the surface load becomes of special interest, thick-plate theory has to be used (see Art. 19).

The moment of the load q and the moment due to change in the force Q_y are neglected in this equation, since they are small quantities of a higher order than those retained. After simplification, Eq. (b) becomes

$$\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + Q_y = 0 \tag{c}$$

In the same manner, by taking moments with respect to the y axis, we obtain

$$\frac{\partial M_{yx}}{\partial y} + \frac{\partial M_x}{\partial x} - Q_x = 0 \tag{d}$$

Since there are no forces in the x and y directions and no moments with respect to the z axis, the three equations (99), (c), and (d) completely define the equilibrium of the element. Let us eliminate the shearing forces Q_x and Q_y from these equations by determining them from Eqs. (c) and (d) and substituting into Eq. (99). In this manner we obtain

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{yx}}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} - \frac{\partial^2 M_{xy}}{\partial x\,\partial y} = -q \tag{e}$$

Observing that $M_{yx} = -M_{xy}$, by virtue of $\tau_{xy} = \tau_{yx}$, we finally represent the equation of equilibrium (e) in the following form:

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} - 2\frac{\partial^2 M_{xy}}{\partial x\,\partial y} = -q \tag{100}$$

To represent this equation in terms of the deflections w of the plate, we make the assumption here that expressions (41) and (43), developed for the case of pure bending, can be used also in the case of laterally loaded plates. This assumption is equivalent to neglecting the effect on bending of the shearing forces Q_x and Q_y and the compressive stress σ_z produced by the load q. We have already used such an assumption in the previous chapter and have seen that the errors in deflections obtained in this way are small provided the thickness of the plate is small in comparison with the dimensions of the plate in its plane. An approximate theory of bending of thin elastic plates, taking into account the effect of shearing forces on the deformation, will be given in Art. 39, and several examples of exact solutions of bending problems of plates will be discussed in Art. 26.

Using x and y directions instead of n and t, which were used in Eqs. (41) and (43), we obtain

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \qquad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \tag{101}$$

$$M_{xy} = -M_{yx} = D(1-\nu)\frac{\partial^2 w}{\partial x\,\partial y} \tag{102}$$

Substituting these expressions in Eq. (100), we obtain[1]

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \tag{103}$$

This latter equation can also be written in the symbolic form

$$\Delta\Delta w = \frac{q}{D} \tag{104}$$

where

$$\Delta w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \tag{105}$$

It is seen that the problem of bending of plates by a lateral load q reduces to the integration of Eq. (103). If, for a particular case, a solution of this equation is found that satisfies the conditions at the boundaries of the plate, the bending and twisting moments can be calculated from Eqs. (101) and (102). The corresponding normal and shearing stresses are found from Eq. (44) and the expression

$$(\tau_{xy})_{\max} = \frac{6M_{xy}}{h^2}$$

Equations (*c*) and (*d*) are used to determine the shearing forces Q_x and Q_y, from which

$$Q_x = \frac{\partial M_{yx}}{\partial y} + \frac{\partial M_x}{\partial x} = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \tag{106}$$

$$Q_y = \frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \tag{107}$$

or, using the symbolic form,

$$Q_x = -D\frac{\partial}{\partial x}(\Delta w) \qquad Q_y = -D\frac{\partial}{\partial y}(\Delta w) \tag{108}$$

The shearing stresses τ_{xz} and τ_{yz} can now be determined by assuming that they are distributed across the thickness of the plate according to the parabolic law.[2] Then

$$(\tau_{xz})_{\max} = \frac{3}{2}\frac{Q_x}{h} \qquad (\tau_{yz})_{\max} = \frac{3}{2}\frac{Q_y}{h}$$

[1] This equation was obtained by Lagrange in 1811, when he was examining the memoir presented to the French Academy of Science by Sophie Germain. The history of the development of this equation is given in I. Todhunter and K. Pearson, "History of the Theory of Elasticity," vol. 1, pp. 147, 247, 348, and vol. 2, part 1, p. 263. See also the note by Saint Venant to Art. 73 on page 689 of the French translation of "Théorie de l'élasticité des corps solides," by Clebsch, Paris, 1883.

[2] It will be shown in Art. 26 that in certain cases this assumption is in agreement with the exact theory of bending of plates.

It is seen that the stresses in a plate can be calculated provided the deflection surface for a given load distribution and for given boundary conditions is determined by integration of Eq. (103).

22. Boundary Conditions. We begin the discussion of boundary conditions with the case of a rectangular plate and assume that the x and y axes are taken parallel to the sides of the plate.

Built-in Edge. If the edge of a plate is built in, the deflection along this edge is zero, and the tangent plane to the deflected middle surface along this edge coincides with the initial position of the middle plane of the plate. Assuming the built-in edge to be given by $x = a$, the boundary conditions are

$$(w)_{x=a} = 0 \qquad \left(\frac{\partial w}{\partial x}\right)_{x=a} = 0 \tag{109}$$

Simply Supported Edge. If the edge $x = a$ of the plate is simply supported, the deflection w along this edge must be zero. At the same time this edge can rotate freely with respect to the edge line; *i.e.*, there are no bending moments M_x along this edge. This kind of support is represented in Fig. 49. The analytical expressions for the boundary conditions in this case are

$$(w)_{x=a} = 0 \qquad \left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)_{x=a} = 0 \tag{110}$$

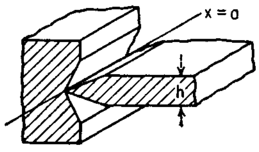

FIG. 49

Observing that $\partial^2 w/\partial y^2$ must vanish together with w along the rectilinear edge $x = a$, we find that the second of the conditions (110) can be rewritten as $\partial^2 w/\partial x^2 = 0$ or also $\Delta w = 0$. Equations (110) are therefore equivalent to the equations

$$(w)_{x=a} = 0 \qquad (\Delta w)_{x=a} = 0 \tag{111}$$

which do not involve Poisson's ratio ν.

Free Edge. If an edge of a plate, say the edge $x = a$ (Fig. 50), is entirely free, it is natural to assume that along this edge there are no bending and twisting moments and also no vertical shearing forces, *i.e.*, that

$$(M_x)_{x=a} = 0 \qquad (M_{xy})_{x=a} = 0 \qquad (Q_x)_{x=a} = 0$$

The boundary conditions for a free edge were expressed by Poisson[1] in this form. But later on, Kirchhoff[2] proved that three boundary conditions are too many and that two conditions are sufficient for the complete determination of the deflections w satisfying Eq. (103). He showed

[1] See the discussion of this subject in Todhunter and Pearson, *op. cit.*, vol. 1, p. 250, and in Saint Venant, *loc. cit.*

[2] See *J. Crelle*, vol. 40, p. 51, 1850.

also that the two requirements of Poisson dealing with the twisting moment M_{xy} and with the shearing force Q_x must be replaced by one boundary condition. The physical significance of this reduction in the number of boundary conditions has been explained by Kelvin and Tait.[1] These authors point out that the bending of a plate will not be changed if the horizontal forces giving the twisting couple $M_{xy}\,dy$ acting on an element of the length dy of the edge $x = a$ are replaced by two vertical forces of magnitude M_{xy} and dy apart, as shown in Fig. 50. Such a replacement does not change the magnitude of twisting moments and produces only local changes in the stress distribution at the edge of the plate, leaving the stress condition of the rest of the plate unchanged. We have already discussed a particular case of such a transformation of the boundary force system in considering pure bending of a plate to an anticlastic surface (see Art. 11). Proceeding with the foregoing replacement of twisting couples along the edge of the plate and considering two adjacent elements of the edge (Fig. 50), we find that the distribution of twisting moments M_{xy} is statically equivalent to a distribution of shearing forces of the intensity

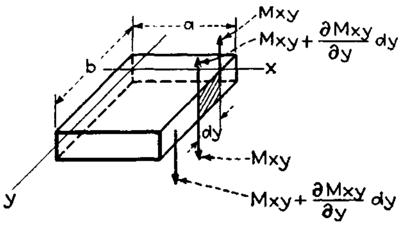

FIG. 50

$$Q'_x = -\left(\frac{\partial M_{xy}}{\partial y}\right)_{x=a}$$

Hence the joint requirement regarding twisting moment M_{xy} and shearing force Q_x along the free edge $x = a$ becomes

$$V_x = \left(Q_x - \frac{\partial M_{xy}}{\partial y}\right)_{x=a} = 0 \qquad (a)$$

Substituting for Q_x and M_{xy} their expressions (106) and (102), we finally obtain for a free edge $x = a$:

$$\left[\frac{\partial^3 w}{\partial x^3} + (2 - \nu)\frac{\partial^3 w}{\partial x\,\partial y^2}\right]_{x=a} = 0 \qquad (112)$$

The condition that bending moments along the free edge are zero requires

$$\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)_{x=a} = 0 \qquad (113)$$

[1] See "Treatise of Natural Philosophy," vol. 1, part 2, p. 188, 1883. Independently the same question was explained by Boussinesq, *J. Math.*, ser. 2, vol. 16, pp. 125–274, 1871; ser. 3, vol. 5, pp. 329–344, Paris, 1879.

Equations (112) and (113) represent the two necessary boundary conditions along the free edge $x = a$ of the plate.

Transforming the twisting couples as explained in the foregoing discussion and as shown in Fig. 50, we obtain not only shearing forces Q'_x distributed along the edge $x = a$ but also two concentrated forces at the ends of that edge, as indicated in Fig. 51. The magnitudes of these forces are equal to the magnitudes of the twisting couple[1] M_{xy} at the corresponding corners of the plate. Making the analogous transformation of twisting couples M_{yx} along the edge $y = b$, we shall find that in this case again, in addition to the distributed shearing forces Q'_y, there will be concentrated forces M_{yx} at the corners. This indicates that a rectangular plate supported in some way along the edges and loaded laterally will usually produce not only reactions distributed along the boundary but also concentrated reactions at the corners.

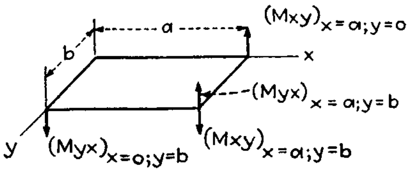

FIG. 51

Regarding the directions of these concentrated reactions, a conclusion can be drawn if the general shape of the deflection surface is known. Take, for example, a uniformly loaded square plate simply supported along the edges. The general shape of the deflection surface is indicated in Fig. 52a by dashed lines representing the section of the middle surface of the plate by planes parallel to the xz and yz coordinate planes. Considering these lines, it may be seen that near the corner A the derivative $\partial w/\partial x$, representing the slope of the deflection surface in the x direction, is negative and decreases numerically with increasing y. Hence $\partial^2 w/\partial x\, \partial y$ is positive at the corner A. From Eq. (102) we conclude that M_{xy} is positive and M_{yx} is negative at that corner. From this and from the directions of M_{xy} and M_{yx} in Fig. 48a it follows that both concentrated forces, indicated at the point $x = a$, $y = b$ in Fig. 51, have a downward direction. From symmetry we conclude also that the forces have the same magnitude and direction at all corners of the plate. Hence the conditions are as indicated in Fig. 52b, in which

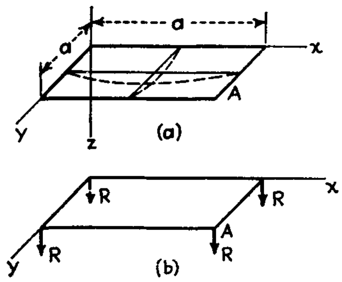

FIG. 52

$$R = 2(M_{xy})_{x=a,y=a} = 2D(1 - \nu)\left(\frac{\partial^2 w}{\partial x\, \partial y}\right)_{x=a,y=a}$$

[1] The couple M_{xy} is a moment per unit length and has the dimension of a force.

It can be seen that, when a square plate is uniformly loaded, the corners in general have a tendency to rise, and this is prevented by the concentrated reactions at the corners, as indicated in the figure.

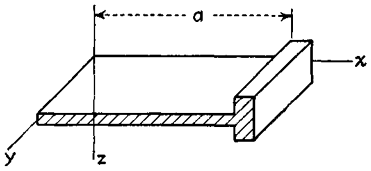

FIG. 53

Elastically Supported and Elastically Built-in Edge. If the edge $x = a$ of a rectangular plate is rigidly joined to a supporting beam (Fig. 53), the deflection along this edge is not zero and is equal to the deflection of the beam. Also, rotation of the edge is equal to the twisting of the beam. Let B be the flexural and C the torsional rigidity of the beam. The pressure in the z direction transmitted from the plate to the supporting beam, from Eq. (a), is

$$-V_x = -\left(Q_x - \frac{\partial M_{xy}}{\partial y}\right)_{x=a} = D \frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial x^2} + (2 - \nu)\frac{\partial^2 w}{\partial y^2}\right]_{x=a}$$

and the differential equation of the deflection curve of the beam is

$$B\left(\frac{\partial^4 w}{\partial y^4}\right)_{x=a} = D \frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial x^2} + (2 - \nu)\frac{\partial^2 w}{\partial y^2}\right]_{x=a} \tag{114}$$

This equation represents one of the two boundary conditions of the plate along the edge $x = a$.

To obtain the second condition, the twisting of the beam should be considered. The angle of rotation[1] of any cross section of the beam is $-(\partial w/\partial x)_{x=a}$, and the rate of change of this angle along the edge is

$$-\left(\frac{\partial^2 w}{\partial x \, \partial y}\right)_{x=a}$$

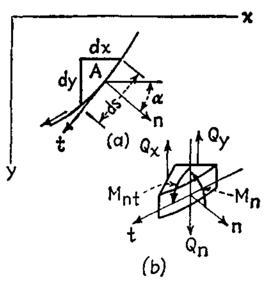

FIG. 54

Hence the twisting moment in the beam is $-C(\partial^2 w/\partial x \, \partial y)_{x=a}$. This moment varies along the edge, since the plate, rigidly connected with the beam, transmits continuously distributed twisting moments to the beam. The magnitude of these applied moments per unit length is equal and opposite to the bending moments M_x in the plate. Hence, from a consideration of the rotational equilibrium of an element of the beam, we obtain

$$-C \frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x \, \partial y}\right)_{x=a} = -(M_x)_{x=a}$$

[1] The right-hand-screw rule is used for the sign of the angle.

or, substituting for M_x its expression (101),

$$-C\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)_{x=a} = D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)_{x=a} \tag{115}$$

This is the second boundary condition at the edge $x = a$ of the plate.

In the case of a plate with a curvilinear boundary (Fig. 54), we take at a point A of the edge the coordinate axes in the direction of the tangent t and the normal n as shown in the figure. The bending and twisting moments at that point are

$$M_n = \int_{-h/2}^{h/2} z\sigma_n\,dz \qquad M_{nt} = -\int_{-h/2}^{h/2} z\tau_{nt}\,dz \tag{b}$$

Using for the stress components σ_n and τ_{nt} the known expressions[1]

$$\sigma_n = \sigma_x\cos^2\alpha + \sigma_y\sin^2\alpha + 2\tau_{xy}\sin\alpha\cos\alpha$$
$$\tau_{nt} = \tau_{xy}(\cos^2\alpha - \sin^2\alpha) + (\sigma_y - \sigma_x)\sin\alpha\cos\alpha$$

we can represent expressions (b) in the following form:

$$\begin{aligned} M_n &= M_x\cos^2\alpha + M_y\sin^2\alpha - 2M_{xy}\sin\alpha\cos\alpha \\ M_{nt} &= M_{xy}(\cos^2\alpha - \sin^2\alpha) + (M_x - M_y)\sin\alpha\cos\alpha \end{aligned} \tag{c}$$

The shearing force Q_n at point A of the boundary will be found from the equation of equilibrium of an element of the plate shown in Fig. 54b, from which

$$Q_n\,ds = Q_x\,dy - Q_y\,dx$$

or

$$Q_n = Q_x\cos\alpha + Q_y\sin\alpha \tag{d}$$

Having expressions (c) and (d), the boundary condition in each particular case can be written without difficulty.

If the curvilinear edge of the plate is built in, we have for such an edge

$$w = 0 \qquad \frac{\partial w}{\partial n} = 0 \tag{e}$$

In the case of a simply supported edge we have

$$w = 0 \qquad M_n = 0 \tag{f}$$

Substituting for M_n its expression from the first of equations (c) and using Eqs. (101) and (102), we can represent the boundary conditions (f) in terms of w and its derivatives.

If the edge of a plate is free, the boundary conditions are

$$M_n = 0 \qquad V_n = Q_n - \frac{\partial M_{nt}}{\partial s} = 0 \tag{g}$$

[1] The x and y directions are not the principal directions as in the case of pure bending; hence the expressions for M_n and M_{nt} will be different from those given by Eqs. (39) and (40)

where the term $-\partial M_{nt}/\partial s$ is obtained in the manner shown in Fig. 50 and represents the portion of the edge reaction which is due to the distribution along the edge of the twisting moment M_{nt}. Substituting expressions (*c*) and (*d*) for M_n, M_{nt}, and Q_n and using Eqs. (101), (102), (106), and (107), we can represent boundary conditions (*g*) in the following form:

$$\nu\,\Delta w + (1-\nu)\left(\cos^2\alpha\,\frac{\partial^2 w}{\partial x^2} + \sin^2\alpha\,\frac{\partial^2 w}{\partial y^2} + \sin 2\alpha\,\frac{\partial^2 w}{\partial x\,\partial y}\right) = 0$$

$$\cos\alpha\,\frac{\partial}{\partial x}\Delta w + \sin\alpha\,\frac{\partial}{\partial y}\Delta w + (1-\nu)\frac{\partial}{\partial s}\left[\cos 2\alpha\,\frac{\partial^2 w}{\partial x\,\partial y} + \frac{1}{2}\sin 2\alpha\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2}\right)\right] = 0 \tag{116}$$

where, as before,

$$\Delta w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$

Another method of derivation of these conditions will be shown in the next article.

23. Alternative Method of Derivation of the Boundary Conditions. The differential equation (104) of the deflection surface of a plate and the boundary conditions can be obtained by using the principle of virtual displacements together with the expression for the strain energy of a bent plate.[1] Since the effect of shearing stress on the deflections was entirely neglected in the derivation of Eq. (104), the corresponding expression for the strain energy will contain only terms depending on the action of bending and twisting moments as in the case of pure bending discussed in Art. 12. Using Eq. (48) we obtain for the strain energy in an infinitesimal element

$$dV = \frac{1}{2}D\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]\right\}dx\,dy \tag{a}$$

The total strain energy of the plate is then obtained by integration as follows:

$$V = \frac{1}{2}D\iint\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]\right\}dx\,dy \tag{117}$$

where the integration is extended over the entire surface of the plate.

Applying the principle of virtual displacements, we assume that an infinitely small variation δw of the deflections w of the plate is produced. Then the corresponding change in the strain energy of the plate must be equal to the work done by the external forces during the assumed virtual displacement. In calculating this work we must consider not only the lateral load q distributed over the surface of the plate but also the bending moments M_n and transverse forces $Q_n - (\partial M_{nt}/\partial s)$ distributed along the boundary of the plate. Hence the general equation, given by the principle of virtual displacements, is

[1] This is the method by which the boundary conditions were satisfactorily established for the first time; see G. Kirchhoff in *J. Crelle*, vol. 40, 1850, and also his Vorlesungen über Mathematische Physik, *Mechanik*, p. 450, 1877. Lord Kelvin took an interest in Kirchhoff's derivations and spoke with Helmholtz about them; see the biography of Kelvin by Sylvanus Thompson, vol. 1, p. 432.