CHAPTER 2

PURE BENDING OF PLATES

9. Slope and Curvature of Slightly Bent Plates. In discussing small deflections of a plate we take the *middle plane* of the plate, before bending occurs, as the xy plane. During bending, the particles that were in the xy plane undergo small displacements w perpendicular to the xy plane and form the *middle surface* of the plate. These displacements of the middle surface are called *deflections* of a plate in our further discussion. Taking a normal section of the plate parallel to the xz plane (Fig. 16*a*), we find that the slope of the middle surface in the x direction is $i_x = \partial w/\partial x$. In the same manner the slope in the y direction is $i_y = \partial w/\partial y$. Taking now any direction an in the xy plane (Fig. 16*b*) making an angle α with the x axis, we find that the difference in the deflections of the two adjacent points a and a_1 in the an direction is

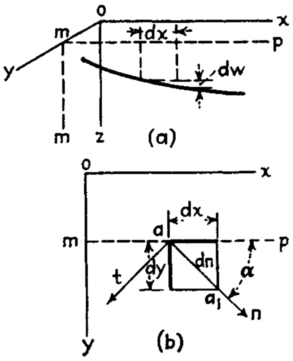

FIG. 16

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy$$

and that the corresponding slope is

$$\frac{\partial w}{\partial n} = \frac{\partial w}{\partial x}\frac{dx}{dn} + \frac{\partial w}{\partial y}\frac{dy}{dn} = \frac{\partial w}{\partial x}\cos\alpha + \frac{\partial w}{\partial y}\sin\alpha \tag{a}$$

To find the direction α_1 for which the slope is a maximum we equate to zero the derivative with respect to α of expression (*a*) In this way we obtain

$$\tan\alpha_1 = \frac{\partial w}{\partial y}\Big/\frac{\partial w}{\partial x} \tag{b}$$

Substituting the corresponding values of $\sin\alpha_1$ and $\cos\alpha_1$ in (*a*), we obtain for the maximum slope the expression

$$\left(\frac{\partial w}{\partial n}\right)_{max} = \sqrt{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2} \tag{c}$$

By setting expression (*a*) equal to zero we obtain the direction for which the slope of the surface is zero. The corresponding angle α_2 is determined from the equation

$$\tan \alpha_2 = -\frac{\partial w}{\partial x} \bigg/ \frac{\partial w}{\partial y} \tag{d}$$

From Eqs. (*b*) and (*d*) we conclude that

$$\tan \alpha_1 \tan \alpha_2 = -1$$

which shows that the directions of zero slope and of maximum slope are perpendicular to each other.

In determining the curvature of the middle surface of the plate we observe that the deflections of the plate are very small. In such a case the slope of the surface in any direction can be taken equal to the angle that the tangent to the surface in that direction makes with the xy plane, and the square of the slope may be neglected compared to unity. The curvature of the surface in a plane parallel to the xz plane (Fig. 16) is then numerically equal to

$$\frac{1}{r_x} = -\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = -\frac{\partial^2 w}{\partial x^2} \tag{e}$$

We consider a curvature positive if it is convex downward. The minus sign is taken in Eq. (*e*), since for the deflection convex downward, as shown in the figure, the second derivative $\partial^2 w/\partial x^2$ is negative.

In the same manner we obtain for the curvature in a plane parallel to the yz plane

$$\frac{1}{r_y} = -\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) = -\frac{\partial^2 w}{\partial y^2} \tag{f}$$

These expressions are similar to those used in discussing the curvature of a bent beam.

In considering the curvature of the middle surface in any direction an (Fig. 16) we obtain

$$\frac{1}{r_n} = -\frac{\partial}{\partial n}\left(\frac{\partial w}{\partial n}\right)$$

Substituting expression (*a*) for $\partial w/\partial n$ and observing that

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial x}\cos\alpha + \frac{\partial}{\partial y}\sin\alpha$$

we find

$$\begin{aligned}\frac{1}{r_n} &= -\left(\frac{\partial}{\partial x}\cos\alpha + \frac{\partial}{\partial y}\sin\alpha\right)\left(\frac{\partial w}{\partial x}\cos\alpha + \frac{\partial w}{\partial y}\sin\alpha\right)\\ &= -\left(\frac{\partial^2 w}{\partial x^2}\cos^2\alpha + 2\frac{\partial^2 w}{\partial x\,\partial y}\sin\alpha\cos\alpha + \frac{\partial^2 w}{\partial y^2}\sin^2\alpha\right)\\ &= \frac{1}{r_x}\cos^2\alpha - \frac{1}{r_{xy}}\sin 2\alpha + \frac{1}{r_y}\sin^2\alpha\end{aligned} \tag{g}$$

It is seen that the curvature in any direction n at a point of the middle surface can be calculated if we know at that point the curvatures

$$\frac{1}{r_x} = -\frac{\partial^2 w}{\partial x^2} \qquad \frac{1}{r_y} = -\frac{\partial^2 w}{\partial y^2}$$

and the quantity

$$\frac{1}{r_{xy}} = \frac{\partial^2 w}{\partial x\,\partial y} \tag{h}$$

which is called the *twist of the surface* with respect to the x and y axes.

If instead of the direction an (Fig. 16*b*) we take the direction at perpendicular to an, the curvature in this new direction will be obtained from expression (g) by substituting $\pi/2 + \alpha$ for α. Thus we obtain

$$\frac{1}{r_t} = \frac{1}{r_x}\sin^2\alpha + \frac{1}{r_{xy}}\sin 2\alpha + \frac{1}{r_y}\cos^2\alpha \tag{i}$$

Adding expressions (g) and (i), we find

$$\frac{1}{r_n} + \frac{1}{r_t} = \frac{1}{r_x} + \frac{1}{r_y} \tag{34}$$

which shows that at any point of the middle surface the sum of the curvatures in two perpendicular directions such as n and t is independent of the angle α. This sum is usually called the *average curvature* of the surface at a point.

The twist of the surface at a with respect to the an and at directions is

$$\frac{1}{r_{nt}} = \frac{d}{dt}\left(\frac{dw}{dn}\right)$$

In calculating the derivative with respect to t, we observe that the direction at is perpendicular to an. Thus we obtain the required derivative by substituting $\pi/2 + \alpha$ for α in Eq. (a). In this manner we find

$$\begin{aligned}\frac{1}{r_{nt}} &= \left(\frac{\partial}{\partial x}\cos\alpha + \frac{\partial}{\partial y}\sin\alpha\right)\left(-\frac{\partial w}{\partial x}\sin\alpha + \frac{\partial w}{\partial y}\cos\alpha\right)\\ &= \frac{1}{2}\sin 2\alpha\left(-\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \cos 2\alpha\,\frac{\partial^2 w}{\partial x\,\partial y}\\ &= \frac{1}{2}\sin 2\alpha\left(\frac{1}{r_x} - \frac{1}{r_y}\right) + \cos 2\alpha\,\frac{1}{r_{xy}}\end{aligned} \tag{j}$$

In our further discussion we shall be interested in finding in terms of α the directions in which the curvature of the surface is a maximum or a minimum and in finding the corresponding values of the curvature. We obtain the necessary equation for determining α by equating the derivative of expression (g) with respect to α to zero, which gives

$$\frac{1}{r_x}\sin 2\alpha + \frac{2}{r_{xy}}\cos 2\alpha - \frac{1}{r_y}\sin 2\alpha = 0 \tag{k}$$

whence

$$\tan 2\alpha = -\frac{\dfrac{2}{r_{xy}}}{\dfrac{1}{r_x} - \dfrac{1}{r_y}} \tag{35}$$

From this equation we find two values of α, differing by $\pi/2$. Substituting these in Eq. (g) we find two values of $1/r_n$, one representing the maximum and the other the minimum curvature at a point a of the surface. These two curvatures are called the *principal curvatures* of the surface; and the corresponding planes naz and taz, the *principal planes of curvature.*

Observing that the left-hand side of Eq. (k) is equal to the doubled value of expression (j), we conclude that, if the directions an and at (Fig. 16) are in the principal planes, the corresponding twist $1/r_{nt}$ is equal to zero.

We can use a circle, similar to Mohr's circle representing combined stresses, to show how the curvature and the twist of a surface vary with the angle α.* To simplify the discussion we assume that the coordinate planes xz and yz are taken parallel to the principal planes of curvature at the point a. Then

$$\frac{1}{r_{xy}} = 0$$

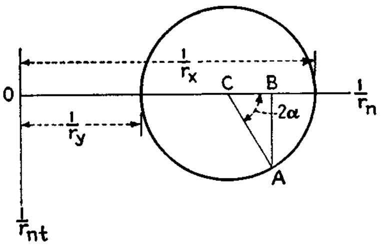

Fig. 17

and we obtain from Eqs. (g) and (j) for any angle α

$$\begin{aligned} \frac{1}{r_n} &= \frac{1}{r_x}\cos^2\alpha + \frac{1}{r_y}\sin^2\alpha \\ \frac{1}{r_{nt}} &= \frac{1}{2}\left(\frac{1}{r_x} - \frac{1}{r_y}\right)\sin 2\alpha \end{aligned} \tag{36}$$

Taking the curvatures as abscissas and the twists as ordinates and constructing a circle on the diameter $1/r_x - 1/r_y$, as shown in Fig. 17, we see that the point A defined by the angle 2α has the abscissa

$$\begin{aligned} \overline{OB} = \overline{OC} + \overline{CB} &= \frac{1}{2}\left(\frac{1}{r_x} + \frac{1}{r_y}\right) + \frac{1}{2}\left(\frac{1}{r_x} - \frac{1}{r_y}\right)\cos 2\alpha \\ &= \frac{1}{r_x}\cos^2\alpha + \frac{1}{r_y}\sin^2\alpha \end{aligned}$$

and the ordinate

$$\overline{AB} = \frac{1}{2}\left(\frac{1}{r_x} - \frac{1}{r_y}\right)\sin 2\alpha$$

Comparing these results with formulas (36), we conclude that the coordi-

* See S. Timoshenko, "Strength of Materials," part I, 3d ed., p. 40, 1955.

nates of the point A define the curvature and the twist of the surface for any value of the angle α. It is seen that the maximum twist, represented by the radius of the circle, takes place when $\alpha = \pi/4$, *i.e.*, when we take two perpendicular directions bisecting the angles between the principal planes.

In our example the curvature in any direction is positive; hence the surface is bent convex downward. If the curvatures $1/r_x$ and $1/r_y$ are both negative, the curvature in any direction is also negative, and we have a bending of the plate convex upward. Surfaces in which the curvatures in all planes have like signs are called *synclastic.* Sometimes we shall deal with surfaces in which the two principal curvatures have opposite signs. A saddle is a good example. Such surfaces are called *anticlastic.* The circle in Fig. 18 represents a particular case of such surfaces when

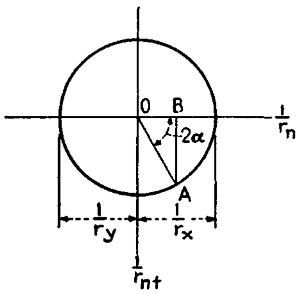

Fig. 18

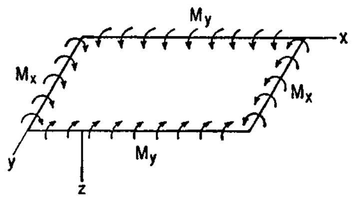

Fig. 19

$1/r_y = -1/r_x$. It is seen that in this case the curvature becomes zero for $\alpha = \pi/4$ and for $\alpha = 3\pi/4$, and the twist becomes equal to $\pm 1/r_x$.

10. Relations between Bending Moments and Curvature in Pure Bending of Plates. In the case of pure bending of prismatic bars a rigorous solution for stress distribution is obtained by assuming that cross sections of the bar remain plane during bending and rotate only with respect to their neutral axes so as to be always normal to the deflection curve. Combination of such bending in two perpendicular directions brings us to pure bending of plates. Let us begin with pure bending of a rectangular plate by moments that are uniformly distributed along the edges of the plate, as shown in Fig. 19. We take the xy plane to coincide with the middle plane of the plate before deflection and the x and y axes along the edges of the plate as shown. The z axis, which is then perpendicular to the middle plane, is taken positive downward. We denote by M_x the bending moment per unit length acting on the edges parallel to the y axis and by M_y the moment per unit length acting on the edges parallel to the x axis. These moments we consider positive when they are directed as shown in the figure, *i.e.*, when they produce compression in the upper surface of the plate and tension in the lower. The thickness of the plate we denote, as before, by h and consider it small in comparison with other dimensions.

Let us consider an element cut out of the plate by two pairs of planes parallel to the xz and yz planes, as shown in Fig. 20. Since the case shown in Fig. 19 represents the combination of two uniform bendings, the stress conditions are identical in all elements, as shown in Fig. 20, and we have a uniform bending of the plate. Assuming that during bending of the plate the lateral sides of the element remain plane and rotate about the neutral axes nn so as to remain normal to the deflected middle surface of the plate, it can be concluded that the middle plane of the plate does not undergo any extension during this bending, and the middle surface is therefore the *neutral surface.*[1] Let $1/r_x$ and $1/r_y$ denote, as before, the curvatures of this neutral surface in sections parallel to the xz and yz planes, respectively. Then the unit elongations in the x and y directions of an elemental lamina $abcd$ (Fig. 20), at a distance z from the neutral surface, are found, as in the case of a beam, and are equal to

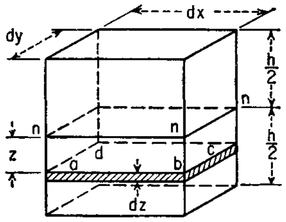

FIG. 20

$$\epsilon_x = \frac{z}{r_x} \qquad \epsilon_y = \frac{z}{r_y} \tag{a}$$

Using now Hooke's law [Eq. (1), page 5], the corresponding stresses in the lamina $abcd$ are

$$\begin{aligned}\sigma_x &= \frac{Ez}{1-\nu^2}\left(\frac{1}{r_x} + \nu\frac{1}{r_y}\right)\\ \sigma_y &= \frac{Ez}{1-\nu^2}\left(\frac{1}{r_y} + \nu\frac{1}{r_x}\right)\end{aligned} \tag{b}$$

These stresses are proportional to the distance z of the lamina $abcd$ from the neutral surface and depend on the magnitude of the curvatures of the bent plate.

The normal stresses distributed over the lateral sides of the element in Fig. 20 can be reduced to couples, the magnitudes of which per unit length evidently must be equal to the external moments M_x and M_y. In this way we obtain the equations

$$\begin{aligned}\int_{-h/2}^{h/2} \sigma_x z\, dy\, dz &= M_x\, dy\\ \int_{-h/2}^{h/2} \sigma_y z\, dx\, dz &= M_y\, dx\end{aligned} \tag{c}$$

[1] It will be shown in Art. 13 that this conclusion is accurate enough if the deflections of the plate are small in comparison with the thickness h.

Substituting expressions (*b*) for σ_x and σ_y, we obtain

$$M_x = D\left(\frac{1}{r_x} + \nu\frac{1}{r_y}\right) = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \tag{37}$$

$$M_y = D\left(\frac{1}{r_y} + \nu\frac{1}{r_x}\right) = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \tag{38}$$

where D is the flexural rigidity of the plate defined by Eq. (3), and w denotes small deflections of the plate in the z direction.

Let us now consider the stresses acting on a section of the lamina *abcd* parallel to the z axis and inclined to the x and y axes. If *acd* (Fig. 21) represents a portion of the lamina cut by such a section, the stress acting on the side *ac* can be found by means of the equations of statics. Resolving this stress into a normal component σ_n and a shearing component τ_{nt},

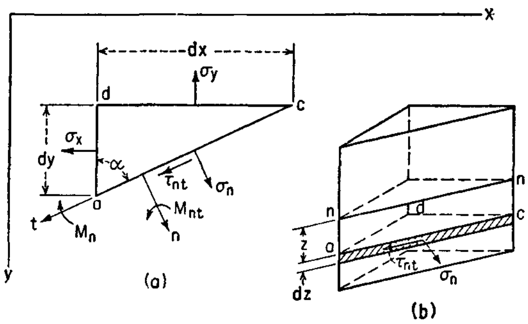

FIG. 21

the magnitudes of these components are obtained by projecting the forces acting on the element *acd* on the n and t directions respectively, which gives the known equations

$$\begin{aligned} \sigma_n &= \sigma_x \cos^2 \alpha + \sigma_y \sin^2 \alpha \\ \tau_{nt} &= \tfrac{1}{2}(\sigma_y - \sigma_x) \sin 2\alpha \end{aligned} \tag{d}$$

in which α is the angle between the normal n and the x axis or between the direction t and the y axis (Fig. 21*a*). The angle is considered positive if measured in a clockwise direction.

Considering all laminas, such as *acd* in Fig. 21*b*, over the thickness of the plate, the normal stresses σ_n give the bending moment acting on the section *ac* of the plate, the magnitude of which per unit length along *ac* is

$$M_n = \int_{-h/2}^{h/2} \sigma_n z\, dz = M_x \cos^2 \alpha + M_y \sin^2 \alpha \tag{39}$$

The shearing stresses τ_{nt} give the twisting moment acting on the section *ac* of the plate, the magnitude of which per unit length of *ac* is

$$M_{nt} = -\int_{-h/2}^{h/2} \tau_{nt} z\, dz = \tfrac{1}{2} \sin 2\alpha (M_x - M_y) \tag{40}$$

The signs of M_n and M_{nt} are chosen in such a manner that the positive values of these moments are represented by vectors in the positive directions of n and t (Fig. 21*a*) if the rule of the right-hand screw is used. When α is zero or π, Eq. (39) gives $M_n = M_x$. For $\alpha = \pi/2$ or $3\pi/2$, we obtain $M_n = M_y$. The moments M_{nt} become zero for these values of α. Thus we obtain the conditions shown in Fig. 19.

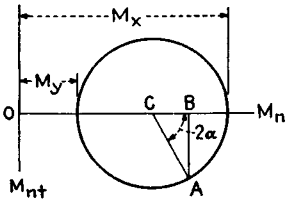

Fig. 22

Equations (39) and (40) are similar to Eqs. (36), and by using them the bending and twisting moments can be readily calculated for any value of α. We can also use the graphical method for the same purpose and find the values of M_n and M_{nt} from Mohr's circle, which can be constructed as shown in the previous article by taking M_n as abscissa and M_{nt} as ordinate. The diameter of the circle will be equal to $M_x - M_y$, as shown in Fig. 22. Then the coordinates $\overline{OB}$ and $\overline{AB}$ of a point A, defined by the angle 2α, give the moments M_n and M_{nt} respectively.

Let us now represent M_n and M_{nt} as functions of the curvatures and twist of the middle surface of the plate. Substituting in Eq. (39) for M_x and M_y their expressions (37) and (38), we find

$$M_n = D\left(\frac{1}{r_x}\cos^2\alpha + \frac{1}{r_y}\sin^2\alpha\right) + \nu D\left(\frac{1}{r_x}\sin^2\alpha + \frac{1}{r_y}\cos^2\alpha\right)$$

Using the first of the equations (36) of the previous article, we conclude that the expressions in parentheses represent the curvatures of the middle surface in the n and t directions respectively. Hence

$$M_n = D\left(\frac{1}{r_n} + \nu\frac{1}{r_t}\right) = -D\left(\frac{\partial^2 w}{\partial n^2} + \nu\frac{\partial^2 w}{\partial t^2}\right) \tag{41}$$

To obtain the corresponding expression for the twisting moment M_{nt}, let us consider the distortion of a thin lamina *abcd* with the sides *ab* and *ad* parallel to the n and t directions and at a distance z from the middle plane (Fig. 23). During bending of the plate the points *a*, *b*, *c*, and *d* undergo small displacements. The components of the displacement of the point a in the n and t directions we denote by u and v respectively. Then the displacement of the adjacent point d in the n direction is $u + (\partial u/\partial t)\, dt$, and the displacement of the point b in the t direction is $v + (\partial v/\partial n)\, dn$. Owing to these displacements, we obtain for the shearing strain

$$\gamma_{nt} = \frac{\partial u}{\partial t} + \frac{\partial v}{\partial n} \tag{e}$$

The corresponding shearing stress is

$$\tau_{nt} = G\left(\frac{\partial u}{\partial t} + \frac{\partial v}{\partial n}\right) \tag{f}$$

From Fig. 23*b*, representing the section of the middle surface made by the normal plane through the n axis, it may be seen that the angle of rotation in the counterclockwise direction of an element pq, which initially was perpendicular to the xy plane, about an axis perpendicular to the nz plane is equal to $-\partial w/\partial n$. Owing to this rotation a point of the

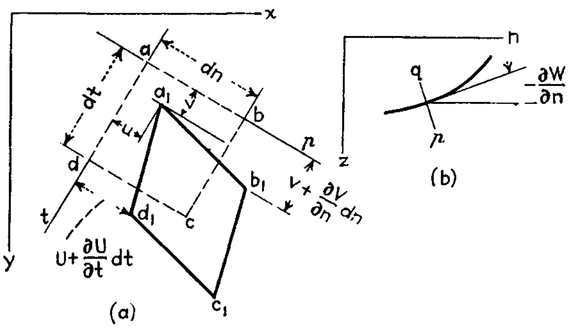

FIG. 23

element at a distance z from the neutral surface has a displacement in the n direction equal to

$$u = -z\frac{\partial w}{\partial n}$$

Considering the normal section through the t axis, it can be shown that the same point has a displacement in the t direction equal to

$$v = -z\frac{\partial w}{\partial t}$$

Substituting these values of the displacements u and v in expression (f), we find

$$\tau_{nt} = -2Gz\frac{\partial^2 w}{\partial n\,\partial t} \tag{42}$$

and expression (40) for the twisting moment becomes

$$M_{nt} = -\int_{-h/2}^{h/2} \tau_{nt} z\,dz = \frac{Gh^3}{6}\frac{\partial^2 w}{\partial n\,\partial t} = D(1-\nu)\frac{\partial^2 w}{\partial n\,\partial t} \tag{43}$$

It is seen that the twisting moment for the given perpendicular directions n and t is proportional to the twist of the middle surface corresponding to those directions. When the n and t directions coincide with the x and y axes, there are only bending moments M_x and M_y acting on the sections perpendicular to those axes (Fig. 19). Hence the corresponding twist is zero, and the curvatures $1/r_x$ and $1/r_y$ are the principal curvatures of the middle surface of the plate. They can readily be calculated from Eqs. (37) and (38) if the bending moments M_x and M_y are given. The curvature in any other direction, defined by an angle α, can then be calculated by using the first of the equations (36), or it can be taken from Fig. 17.

Regarding the stresses in a plate undergoing pure bending, it can be concluded from the first of the equations (d) that the maximum normal stress acts on those sections parallel to the xz or yz planes. The magnitudes of these stresses are obtained from Eqs. (b) by substituting $z = h/2$ and by using Eqs. (37) and (38). In this way we find

$$(\sigma_x)_{\max} = \frac{6M_x}{h^2} \qquad (\sigma_y)_{\max} = \frac{6M_y}{h^2} \tag{44}$$

If these stresses are of opposite sign, the maximum shearing stress acts in the plane bisecting the angle between the xz and yz planes and is equal to

$$\tau_{\max} = \frac{1}{2}(\sigma_x - \sigma_y) = \frac{3(M_x - M_y)}{h^2} \tag{45}$$

If the stresses (44) are of the same sign, the maximum shear acts in the plane bisecting the angle between the xy and xz planes or in that bisecting the angle between the xy and yz planes and is equal to $\frac{1}{2}(\sigma_y)_{\max}$ or $\frac{1}{2}(\sigma_x)_{\max}$, depending on which of the two principal stresses $(\sigma_y)_{\max}$ or $(\sigma_x)_{\max}$ is greater.

11. Particular Cases of Pure Bending. In the discussion of the previous article we started with the case of a rectangular plate with uniformly distributed bending moments acting along the edges. To obtain a general case of pure bending of a plate, let us imagine that a portion of any shape is cut out from the plate considered above (Fig. 19) by a cylindrical or prismatic surface perpendicular to the plate. The conditions of bending of this portion will remain unchanged provided that bending and twisting moments that satisfy Eqs. (39) and (40) are distributed along the boundary of the isolated portion of the plate. Thus we arrive at the case of pure bending of a plate of any shape, and we conclude that pure bending is always produced if along the edges of the plate bending moments M_n and twisting moments M_{nt} are distributed in the manner given by Eqs. (39) and (40).

Let us take, as a first example, the particular case in which

$$M_x = M_y = M$$

It can be concluded, from Eqs. (39) and (40), that in this case, for a plate of any shape, the bending moments are uniformly distributed along the entire boundary and the twisting moments vanish. From Eqs. (37) and (38) we conclude that

$$\frac{1}{r_x} = \frac{1}{r_y} = \frac{M}{D(1+\nu)} \tag{46}$$

i.e., the plate in this case is bent to a spherical surface the curvature of which is given by Eq. (46).

In the general case, when M_x is different from M_y, we put

$$M_x = M_1 \qquad \text{and} \qquad M_y = M_2$$

Then, from Eqs. (37) and (38), we find

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} &= -\frac{M_1 - \nu M_2}{D(1-\nu^2)} \\ \frac{\partial^2 w}{\partial y^2} &= -\frac{M_2 - \nu M_1}{D(1-\nu^2)} \end{aligned} \tag{a}$$

and in addition

$$\frac{\partial^2 w}{\partial x\, \partial y} = 0 \tag{b}$$

Integrating these equations, we find

$$w = -\frac{M_1 - \nu M_2}{2D(1-\nu^2)} x^2 - \frac{M_2 - \nu M_1}{2D(1-\nu^2)} y^2 + C_1 x + C_2 y + C_3 \tag{c}$$

where C_1, C_2, and C_3 are constants of integration. These constants define the plane from which the deflections w are measured. If this plane is taken tangent to the middle surface of the plate at the origin, the constants of integration must be equal to zero, and the deflection surface is given by the equation

$$w = -\frac{M_1 - \nu M_2}{2D(1-\nu^2)} x^2 - \frac{M_2 - \nu M_1}{2D(1-\nu^2)} y^2 \tag{d}$$

In the particular case where $M_1 = M_2 = M$, we get from Eq. (*d*)

$$w = -\frac{M(x^2 + y^2)}{2D(1+\nu)} \tag{e}$$

i.e., a paraboloid of revolution instead of the spherical surface given by Eq. (46). The inconsistency of these results arises merely from the use of the approximate expressions $\partial^2 w/\partial x^2$ and $\partial^2 w/\partial y^2$ for the curvatures $1/r_x$ and $1/r_y$ in deriving Eq. (*e*). These second derivatives of the deflections, rather than the exact expressions for the curvatures, will be used also in all fuıther considerations, in accordance with the assumptions made in Art. 9. This procedure greatly simplifies the fundamental equations of the theory of plates.

Returning now to Eq. (*d*), let us put $M_2 = -M_1$. In this case the principal curvatures, from Eqs. (*a*), are

$$\frac{1}{r_x} = -\frac{1}{r_y} = -\frac{\partial^2 w}{\partial x^2} = \frac{M_1}{D(1-\nu)} \tag{f}$$

and we obtain an anticlastic surface the equation of which is

$$w = -\frac{M_1}{2D(1-\nu)}(x^2 - y^2) \tag{g}$$

Straight lines parallel to the x axis become, after bending, parabolic curves convex downward (Fig. 24), whereas straight lines in the y direction become parabolas convex upward. Along the lines bisecting the angles between the x and y axes we have $x = y$, or $x = -y$; thus deflections along these lines, as seen from Eq. (*g*), are zero. All lines parallel to these bisecting lines before bending remain straight during bending, rotating only by some angle. A rectangle *abcd* bounded by such lines will be twisted as shown in Fig. 24. Imagine normal sections of the plate along lines *ab*, *bc*, *cd*, and *ad*. From Eqs. (39) and (40) we conclude that bending moments along these sections are zero and that twisting moments along sections *ad* and *bc* are equal to M_1 and along sections *ab* and *cd* are equal to $-M_1$. Thus the portion *abcd* of the plate is in the condition of a plate undergoing pure bending produced by twisting moments uniformly distributed along the edges (Fig. 25*a*). These twisting moments are formed by the horizontal shearing stresses continuously distributed over the edge [Eq. (40)]. This horizontal stress distribution can be replaced by vertical shearing forces which produce the same effect as the actual distribution of stresses. To show this, let the edge *ab* be divided into infinitely narrow rectangles, such as *mnpq* in Fig. 25*b*. If Δ is the small width of the rectangle, the corresponding twisting couple is $M_1\Delta$ and can be formed by two vertical forces equal to M_1 acting along the vertical sides of the rectangle. This replacement of the distributed horizontal forces by a statically equivalent system of two vertical forces cannot cause any sensible disturbance in the plate, except within a distance comparable with the thickness of the plate,[1] which is assumed small. Proceeding in the same manner with all the rectangles, we find that all forces M_1 acting along the vertical sides of the rectangles balance one another and only two forces M_1 at the corners *a* and *d* are left. Making

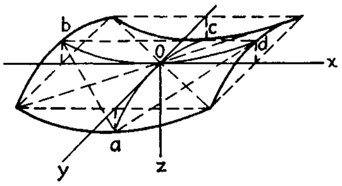

FIG. 24

[1] This follows from *Saint Venant's principle;* see S. Timoshenko and J. N. Goodier, "Theory of Elasticity," 2d ed., p. 33, 1951.

the same transformation along the other edges of the plate, we conclude that bending of the plate to the anticlastic surface shown in Fig. 25*a* can be produced by forces concentrated at the corners[1] (Fig. 25*c*). Such an experiment is comparatively simple to perform, and was used for the experimental verification of the theory of bending of plates discussed above.[2] In these experiments the deflections of the plate along the line *bod* (Fig. 24) were measured and were found to be in very satisfactory agreement with the theoretical results obtained from Eq. (*g*). Some discrepancies were found only near the edges, and they were more pro-

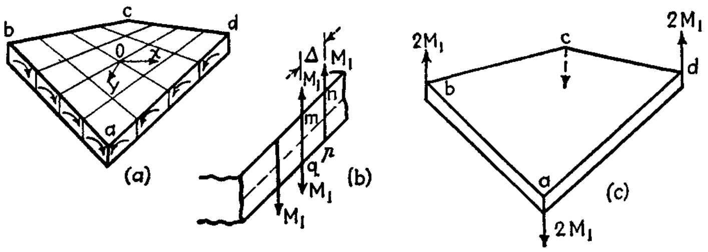

FIG. 25

nounced in the case of comparatively thick plates, as would be expected from the foregoing discussion of the transformation of twisting couples along the edges.

As a last example let us consider the bending of a plate (Fig. 19) to a cylindrical surface having its generating line parallel to the y axis. In such a case $\partial^2 w/\partial y^2 = 0$, and we find, from Eqs. (37) and (38),

$$M_x = -D\frac{\partial^2 w}{\partial x^2} \qquad M_y = -\nu D\frac{\partial^2 w}{\partial x^2} \tag{h}$$

It is seen that to produce bending of the plate to a cylindrical surface we must apply not only the moments M_x but also the moments M_y. Without these latter moments the plate will be bent to an anticlastic surface.[3] The first of equations (*h*) has already been used in Chap. 1 in discussing the bending of long rectangular plates to a cylindrical surface. Although in that discussion we had a bending of plates by lateral loads and there were not only bending stresses but also vertical shearing stresses

[1] This transformation of the force system acting along the edges was first suggested by Lord Kelvin and P. G. Tait; see "Treatise on Natural Philosophy," vol. 1, part 2, p. 203, 1883.

[2] Such experiments were made by A. Nádai, *Forschungsarb.*, vols. 170, 171, Berlin, 1915; see also his book "Elastische Platten," p. 42, Berlin, 1925.

[3] We always assume very small deflections or else bending to a developable surface. The case of bending to a nondevelopable surface when the deflections are not small will be discussed later; see p. 47

acting on sections perpendicular to the x axis, it can be concluded from a comparison with the usual beam theory that the effect of the shearing forces is negligible in the case of thin plates, and the equations developed for the case of pure bending can be used with sufficient accuracy for lateral loading.

12. Strain Energy in Pure Bending of Plates. If a plate is bent by uniformly distributed bending moments M_x and M_y (Fig. 19) so that the xz and yz planes are the principal planes of the deflection surface of the plate, the strain energy stored in an element, such as shown in Fig. 20, is obtained by calculating the work done by the moments $M_x\,dy$ and $M_y\,dx$ on the element during bending of the plate. Since the sides of the element remain plane, the work done by the moments $M_x\,dy$ is obtained by taking half the product of the moment and the angle between the corresponding sides of the element after bending. Since $-\partial^2 w/\partial x^2$ represents the curvature of the plate in the xz plane, the angle corresponding to the moments $M_x\,dy$ is $-(\partial^2 w/\partial x^2)\,dx$, and the work done by these moments is

$$-\frac{1}{2} M_x \frac{\partial^2 w}{\partial x^2}\,dx\,dy$$

An analogous expression is also obtained for the work produced by the moments $M_y\,dx$. Then the total work, equal to the strain energy of the element, is

$$dV = -\frac{1}{2}\left(M_x \frac{\partial^2 w}{\partial x^2} + M_y \frac{\partial^2 w}{\partial y^2}\right) dx\,dy$$

Substituting for the moments their expressions (37) and (38), the strain energy of the elements is represented in the following form:

$$dV = \frac{1}{2} D \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] dx\,dy \tag{a}$$

Since in the case of pure bending the curvature is constant over the entire surface of the plate, the total strain energy of the plate will be obtained if we substitute the area A of the plate for the elementary area $dx\,dy$ in expression (a). Then

$$V = \frac{1}{2} DA \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \tag{47}$$

If the directions x and y do not coincide with the principal planes of curvature, there will act on the sides of the element (Fig. 20) not only the bending moments $M_x\,dy$ and $M_y\,dx$ but also the twisting moments $M_{xy}\,dy$ and $M_{yx}\,dx$. The strain energy due to bending moments is represented by expression (a). In deriving the expression for the strain energy due to twisting moments $M_{xy}\,dy$ we observe that the corresponding angle of twist is equal to the rate of change of the slope $\partial w/\partial y$, as x varies,